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ON THE SOLVABILITY OF LINEAR PDEs IN WEIGHTED SOBOLEV SPACES

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Introduction

It is going to be about a problem which is probably the most primitive in partial differential equations theory, namely to know whether an equation does, or does not, have a solution. In particular, the theory of general elliptic boundary value problems in smooth domains was developed in the second half of 20th century by $Maz'ya$, $I.G.Petrovskii$, $M.I.Vishik$, $Ya.B.Lopantiskii$, $V.A.Kondrat'ev$, $S.Agmon$, $A.Douglis$, $L.Nirenberg$, $M.Schechter$, $J.Necas$, $J.L.Lions$, $E.Magenes$.

Fundamental results in this theory are:

- a priori estimates for the solutions in different function spaces;
- the Fredholm property of the operator corresponding to the boundary value problem;
- regularity assertions of the solutions.

In this work we are interested in strong solutions of a Dirichlet problem for an elliptic linear operator. At this aim, let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. Given any $p \in]1, +\infty[$, a linear uniformly elliptic boundary value problem in non divergence form consists of

$$\begin{cases} Lu := -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} u + au = f \qquad \text{in} \;\; \Omega, \\ u = 0 \quad \text{on} \;\; \partial\Omega, \qquad f \in L^p(\Omega), \end{cases} \tag{1}$$

for the unknown function u defined on Ω.

The *uniform ellipticity* of the operator will be expressed, as usual, by the requirement

$$\exists\, \nu > 0 \;\; : \;\; \sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \; \geq \; \nu |\xi|^2 \quad \text{a.e. in} \;\; \Omega\,, \; \forall \xi \in \mathbb{R}^n. \tag{2}$$

We refer to the problem (1) as the homogeneus Dirichlet problem for the linear operator L and we are interested in *strong* solutions for it.

Namely, a strong solution of (1) is a twice weakly differentiable function, $u \in W^{2,p}(\Omega)$, $p \in]1, +\infty[$, that satisfies the equation $Lu = f$ *almost everywhere* (a.e.) in Ω and assumes the boundary values in the sense of $\overset{\circ}{W}{}^{1,p}(\Omega)$. This concept makes sense for $f \in L^p(\Omega)$ and when the coefficients a_{ij} are measurable functions such that

$$a_{ij} = a_{ji} \in L^\infty(\Omega). \tag{3}$$

A reasonable strong solvability theory of (1) cannot be built up without suitable additional hypotheses on leading coefficients.

Indeed, if a_{ij} are continous functions in $\bar{\Omega}$

$$a_{ij} \in C^0(\bar{\Omega}) \tag{4}$$

a satisfactory theory (known "L^p-theory") exists. It provides solvability and regularity for (1) in Sobolev spaces $W^{2,p}(\Omega)$ for $p > 1$ (see the classical monographs [31], [36], [23]).

Unfortunately, even if Ω is bounded and sufficiently regular, simply assuming (2) - (3) it is not enough to ensure the strong solvability as shown by C. Pucci. For relevant counterexamples we refer to [33], [38], [42]. It is well known that the planar case, $n = 2$, exhibits a remarkable exception of such a situation, as shown by G. Talenti in [48], but just whenever p is 2 or is sufficiently close to 2. The exact range I of admissible values of the parameter p assuring the well-posedness has been recently determined in [2]: it does not depend on p, but just on the value of the ellipticity constant $\nu \leq 1$ of the differential operator L, namely $I := [2(1+\nu^2)^{-1}, 2(1-\nu^2)^{-1}]$. The lower critical exponent of I coincides with the one conjectured by C.Pucci in [40], who also proved that the uniqueness of the solution fails for values of p smaller than it.

The next step of the theory deals with weakening the continuity assumption (4). The motivation is linked to the fact that mathematical modeling of numerous physical and engineering phenomena lead to the boundary value problems for discontinuous parabolic or elliptic operators which require strong solutions.

In the framework of discontinuous coefficients (we refer to [34] for a general survey on the subject), special attention is paid to the so-called *Cordes condition* introduced by H. O. Cordes in the study of Hölder continuity of the solutions to (1). The Cordes condition enabled G.Talenti ([47]) to derive strong solvability in $W^{2,2}(\Omega)$ of the Dirichlet problem for the operator L. Another class of discontinuous coefficients is that introduced by C.Miranda in [35] and formed by functions belonging to the Sobolev space $W^{1,n}(\Omega)$, $((a_{ij})_{x_k} \in L^n(\Omega))$, $n \geq 3$. First generalization in this direction have been carried on, always considering a bounded and sufficiently regular set Ω, assuming that the derivatives belong to some wider spaces. In particular, in [1] the $(a_{ij})_{x_k}$ are in the weak-L^n space, while in [18] they are supposed to be in an appropriate subspace of the classical Morrey space $L^{2p,n-2p}(\Omega)$, where $p \in]1, n/2[$. In [21] the leading coefficients are supposed to be close to functions whose derivatives are in $L^n(\Omega)$. Althought these two types of discontinuity are substantially different, the approaches in studying boundary value problems are unified on the base of elegant Miranda - Talenti inequality which permits an exact computation of the costants appearing in L^2 - a priori bounds (see chapter (1.4) of [34]).

In the development of the L^p - theory, for $p \in]1, +\infty[$ and for any regular enough open subset Ω of $\mathbb{R}^n$, $n \geq 2$, one need to impose certain restrictions on the behaviour of the measurable and bounded leading coefficients. In two pioneer articles of '90s, [19, 20], F.Chiarenza, M.Frasca and P.Longo succeeded to modify the classical methods to obtain L^p

estimates of solutions to (1) which allowed to move from (4) into the conditions that a_{ij} belong to the Sarason class VMO of functions whose integral oscillations over balls shrinking to a point coverges uniformly to zero (see [43]). It turns out to assume a kind of continuity in the average sense instead of pointwise sense. Roughly speaking, the approach goes back to A.Calderon and A.Zygmund and makes use of an explicit representation formula for the second derivatives D^2u of any solutions to (1). Thus, if the coefficients a_{ij} have a "small integral oscillation" (that is, $a_{ij} \in VMO$) then the L^p - norm of D^2u is bounded in term of L^p - norm of f and this holds for any $p \in]1, +\infty[$. Taking into account the fact that VMO contains as proper subsets $C^0(\overline{\Omega})$ and $W^{1,n}(\Omega)$, then the L^p - theory of operators with VMO principal coefficients is a generalization of what was known before 1990 if the domain Ω is bounded in $\mathbb{R}^n$ and $n \geq 3$. This weaking continuity of coefficients, as we note in variuous applications, generates boundary value problems for elliptic equations whose ellipticity is "*disturbed*" in the sense that some *degeneration* or *singularity* appears. This "*bad*" behaviour can be caused by the coefficients of the corresponding differential operator and, near the boundary $\partial\Omega$, it can be deal with two situations:

or may exclude the solvability of the Dirichlet problem in classical no weighted Sobolev spaces;

or the problem is solvable in classical Sobolev spaces but from the behaviour of the coefficients near the boundary $\partial\Omega$ we could deduce the analogous one for the solution (see [45], [58]).

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces ([29], [57], [13]).

We note that the role of a weight function consists in fixing the behaviour at infinity of the functions belonging to the weighted Sobolev space and of their derivatives and near the not regular part of boundary of the domain.

In this framework, we can insert our work. In chapter 1, we deal with introducing the weight functions and their corresponding weighted Sobolev spaces to investigate, first of all, why to choose a weighted Sobolev space instead of classical Sobolev spaces and, after, how to select a certain type of weight functions than the other ones. This choice mainly depends by the necessity to obtain a new Sobolev space also Banach space (see [30]). In this point of view, on a subset Ω di $\mathbb{R}^n$, $n \geq 2$, not necessary bounded, two new classes of weight functions are introduced and their properties are examined:

1. $\mathcal{G}(\Omega)$: this class, introduced yet by M. Troisi in [54], is defined as the union of sets $\mathcal{G}_d(\Omega)$ for any $d \in \mathbb{R}_+$:

   $$\mathcal{G}(\Omega) = \bigcup_{d \in \mathbb{R}_+} \mathcal{G}_d(\Omega),$$

   where $\mathcal{G}_d(\Omega)$ is the class of measurable functions $m : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega\\ |x-y|<d}} \frac{m(x)}{m(y)} < +\infty\,, \tag{5}$$

2. $\mathcal{C}^k(\overline{\Omega})$: this class is defined as the set of the functions $\rho : \overline{\Omega} \to \mathbb{R}_+$ such that $\rho \in C^k(\overline{\Omega})$, $k \in \mathbb{N}_0$, and

$$\sup_{x\in\Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty, \quad \forall\, |\alpha| \le k. \tag{6}$$

We stress the point that $\mathcal{C}^k(\overline{\Omega})$ weight functions are more regular than $\mathcal{G}(\Omega)$ - functions. Althought, $\mathcal{G}(\Omega)$ weights have the favourable property to admit among its members a ***regularization function***, that is a function of the same weight type but also belonging to $C^\infty(\Omega)$, so a more regular function than a $\mathcal{C}^k(\overline{\Omega})$ weight.

Chapters 2 and 3 are devoted to the study of the solvability of the Dirichlet problem:

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega) \\ Lu = f, \;\; f \in L^p_s(\Omega), \end{cases} \tag{7}$$

where Ω is an unbounded and sufficiently regular open subset of $\mathbb{R}^n$ ($n \ge 2$), $p \in \,]1, +\infty[$, L is the uniform elliptic second order linear differential operator defined by

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a\,, \tag{8}$$

with coefficients $a_{ij} = a_{ji} \in L^\infty(\Omega)$, $i, j = 1, \ldots, n$, $s \in \mathbb{R}$, $p \in]1, +\infty[$, $W^{2,p}_s(\Omega)$, $\mathring{W}^{1,p}_s(\Omega)$ and $L^p_s(\Omega)$ suitable weighted Sobolev spaces on Ω.

In particular, we confine the problem to $\mathcal{G}(\Omega)$ - weighted Sobolev space. In detail we assume that:

- in chapter 2, Ω is an unbounded domain of $\mathbb{R}^n$, for any $n \geq 3$;
- in chapter 3, Ω is an unbounded domain of the plane ($n = 2$).

Instead, in chapter 4, we deal with the solvability in $\mathcal{C}^k(\overline{\Omega})$ - weighted Sobolev spaces

$$\begin{cases} u \in W^{2,2}_s(\Omega) \cap \mathring{W}^{1,2}_s(\Omega) \\ Lu = f\,, \quad f \in L^2_s(\Omega)\,, \end{cases} \tag{9}$$

where Ω is an unbounded domain of $\mathbb{R}^n$, for any $n \geq 2$.

In chapter 2, we start with certain a priori estimates for the operator L, obtained by means of the following properties, just introduced in chapter 1:

(I) ***topological isomorphism***:

$$u \longrightarrow \sigma^s u$$

(from $W^{k,p}_s(\Omega)$ to $W^{k,p}(\Omega)$ or from $\mathring{W}^{1,p}_s(\Omega)$ to $\mathring{W}^{k,p}(\Omega)$). It leads to go from weighted spaces to no-weighted spaces and to get their properties.

(II) ***compactness and boundedness***: of multiplying operator

$$u \longrightarrow \beta u \tag{10}$$

defined in a weighted Sobolev space and which takes values in a weighted Lebesgue space.

We recall that when Ω is bounded, the problem of determining a priori bounds has been investigated by several authors under various hypotheses on the leading coefficients. It is worth to mention the results proved in [35], [19], [20], [55] and [56], where the coefficients a_{ij} are required to be discontinuous. If the open set Ω is unbounded, a priori bounds are established in [51] and [9] with analogous assumptions to those required in [35]. In ([14], [10], [11]), under similar hypotheses asked in ([19], [20]), the above estimates are obtained too. Here, we extend some results of [19] and [20] to a $\mathcal{G}(\Omega)$ - weighted case.

Actually, we do that just assuming the following hypotheses, listed below, on the coefficients and on the weight functions:

- a_{ij} (in addition to simmetry and boundedness) locally $VMO(\Omega)$ and at infinity close to certain e_{ij}, belonging to a suitable subset of $VMO(\Omega)$,
- a_i and a having sommability conditions of local character,
- weight function, s-th power of a function $m \in \mathcal{G}(\Omega)$, not bounded at infinity and with derivates of its regularization function having

suitable infinity conditions,we get the following a priori bound:

$$||u||_{W_s^{2,p}(\Omega)} \leq c\Big(||Lu||_{L_s^p(\Omega)} + ||u||_{L^p(\Omega_1)}\Big) \quad \forall\, u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega), \tag{11}$$

where $s \in \mathbb{R}$, Ω is sufficiently regular and Ω_1 is a bounded open subset of Ω. This a priori bound allows to deduce that L is a semi-Fredholm operator, that is it has close range and finite - dimensional kernel, which is an essential property to state the solvability of the problem (7).

We wish to stress that an analogous estimate has been obtained in [12], in a different situation. Indeed, in [12] the open set Ω has singular boundary and the coefficients of the operator L are singular near a subset of $\partial\Omega$. Hence, in [12], the weight function goes to zero on such subset of $\partial\Omega$ and then also the weighted Sobolev spaces are different with respect to those considered in this dissertation.

After this, by a method of continuity along a parameter, using a priori estimate (11) and the topological isomorphism, it is possible taking an advantage of an existence and uniqueness result for the following no-weighted problem (see [11])

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f\,, \quad f \in L^p(\Omega)\,, \end{cases} \tag{12}$$

in order to establish a uniqueness and existence theorem for $\mathcal{G}(\Omega)$ - problem (7) for any $n \geq 3$.

In chapter 3, the solvability of the $\mathcal{G}(\Omega)$ - problem (7) for unbounded domains of the plane is presented. Note that the recent contributions to the $W^{2,p}$ - solvability, $p \in]1, +\infty[$, in domains of $\mathbb{R}^2$, bounded as well unbounded, are collected in [15], [16], [17]. Then, we extend the results of [17] to a weighted case. Indeed, using some results in [17], we show that a priori estimate (11) for the solutions of (7), when Ω is an unbounded $C^{1,1}$ domains of the plane for the solutions, leads to an existence and uniqueness theorem.

In chapter 4, we deal with $\mathcal{C}^k(\overline{\Omega})$ - weighted Sobolev spaces on unbounded domains of $\mathbb{R}^n$, $n \geq 2$. As a main result we describe a weighted and a not-weighted a priori $W^{2,2}$-bound. These are obtained under hypotheses of Miranda's type on the leading coefficients and supposing that their derivatives $(a_{ij})_{x_k}$ belong to a suitable Morrey type space, which is a generalization to unbounded domains of the classical Morrey space. Notice that the existence of the derivatives is of crucial relevance in our analysis, since it allows us to rewrite the operator L in divergence form and to use some known results concerning variational operators. A straightforward consequence of our argument is the following $W^{2,2}$-bound, having the only term $\|Lu\|_{L^2(\Omega)}$ in the right hand side,

$$\|u\|_{W^{2,2}(\Omega)} \leq c\|Lu\|_{L^2(\Omega)} \quad \forall\, u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega), \tag{13}$$

where the dependence of the constant c is explicitly described. This kind of estimate often cannot be obtained when dealing with unbounded domains and clearly immediately takes to the uniqueness of the solution of problem (12) for $p = 2$.

In the framework of unbounded domains, under more regular conditions on the boundary, an analogous a priori bound can be found in [50], where more regular assumptions on the a_{ij} are taken into account. We quote here also the results of [7], where, in the spirit of [21], the leading coefficients are supposed to be close, in a specific sense, to functions whose derivatives are in spaces of Morrey type and have a suitable behaviour at infinity.

We show that the $W^{2,2}$-bound obtained in (13) allows us to extend our result passing to the $\mathcal{C}^2(\overline{\Omega})$ weighted case. Infact, using (13) we get the following $\mathcal{C}^2(\overline{\Omega})$ weighted $W^{2,2}_s$-bound:

$$\|u\|_{W^{2,2}_s(\Omega)} \leq c\|Lu\|_{L^2_s(\Omega)} \quad \forall\, u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega).$$

From this a priori estimate, assuming that the weight function satisfies also conditions at infinity

$$\lim_{|x|\to+\infty} \Big(\rho(x) + \frac{1}{\rho(x)}\Big) = +\infty \quad \text{and} \lim_{|x|\to+\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0,$$

we deduce the solvability of problem (9).

Existence and uniqueness results for similar problems in the weighted case, but with different weight functions and different assumptions on the coefficients have been proved in [22]. Recent results concerning a priori estimates for solutions of the Poisson and heat equations in weighted spaces can be found in [28], where weights of Kondrat'ev type are considered.

As a final remark, looking at results and methods described in the present work, we notice that all presented issues can be seen as extension of classical boundary value problems for uniformly linear elliptic operators by means a weakening of conditions on leading coefficients. Such conditions mainly concern about the behaviour of leading coefficients which is described by means the class VMO. Thus, we can expect that a suitable and calibrated interplay between conditions on coefficients and on the nature of the domain leads to an interesting enlargement of the repertoire of solvability conditions for elliptic problems once new suitable conditions on leading coefficients are explored.

Notation and function spaces

Let G be any Lebesgue measurable subset of $\mathbb{R}^n$ and $\Sigma(G)$ be the collection of all Lebesgue measurable subsets of G.

For $F \in \Sigma(G)$,

- $|F|$ denote the Lebesgue measure of F;
- $\mathfrak{D}(F)$ is the class of restrictions to F of functions $\zeta \in C_\circ^\infty(\mathbb{R}^n)$ with $\bar{F} \cap \operatorname{supp} \zeta \subseteq F$;
- if $X(F)$ is a space of functions defined on F, we denote by $X_{\text{loc}}(F)$ the class of all functions $g : F \to \mathbb{R}$ such that $\zeta\, g \in X(F)$ for any $\zeta \in \mathfrak{D}(F)$.

For any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, we put $B(x,r) = \{y \in \mathbb{R}^n \,:\, |y-x| < r\}$, $B_r = B(0,r)$ and $F(x,r) = F \cap B(x,r)$.

Now let us recall the definitions of the function spaces in which the coefficients of the operator (3.3) will belong to.

For $n \geq 2$, $\lambda \in [0,n[$, $p \in [1,+\infty[$ and fixed t in $\mathbb{R}_+$, the space of

Morrey type $M^{p,\lambda}(\Omega,t)$ is the set of all functions g in $L^p_{loc}(\overline{\Omega})$ such that

$$\|g\|_{M^{p,\lambda}(\Omega,t)} = \sup_{\substack{\tau\in]0,t]\\ x\in\Omega}} \tau^{-\lambda/p}\|g\|_{L^p(\Omega(x,\tau))} < +\infty, \tag{14}$$

endowed with the norm defined in (14). It is easily seen that, for any $t_1, t_2 \in \mathbb{R}_+$, a function g belongs to $M^{p,\lambda}(\Omega,t_1)$ if and only if it belongs to $M^{p,\lambda}(\Omega,t_2)$, moreover the norms of g in these two spaces are equivalent. This allows us to restrict our attention to the space $M^{p,\lambda}(\Omega) = M^{p,\lambda}(\Omega,1)$.

We now introduce three subspaces of $M^{p,\lambda}(\Omega)$ needed in the sequel. The set $VM^{p,\lambda}(\Omega)$ is made up of the functions $g \in M^{p,\lambda}(\Omega)$ such that

$$\lim_{t\to 0} \|g\|_{M^{p,\lambda}(\Omega,t)} = 0,$$

while $\tilde{M}^{p,\lambda}(\Omega)$ and $M^{p,\lambda}_{\circ}(\Omega)$ denote the closures of $L^\infty(\Omega)$ and $C^\infty_\circ(\Omega)$ in $M^{p,\lambda}(\Omega)$, respectively. We point out that

$$M^{p,\lambda}_{\circ}(\Omega) \subset \tilde{M}^{p,\lambda}(\Omega) \subset VM^{p,\lambda}(\Omega).$$

We put $M^p(\Omega) = M^{p,0}(\Omega)$, $VM^p(\Omega) = VM^{p,0}(\Omega)$, $\tilde{M}^p(\Omega) = \tilde{M}^{p,0}(\Omega)$ and $M^p_\circ(\Omega) = M^{p,0}_\circ(\Omega)$. Hence, one can consider the subset $M^p(\Omega)$ of $L^p_{\mathrm{loc}}(\bar{\Omega})$ consisting of those functions g such that

$$||g||_{M^p(\Omega)} = \sup_{x\in\Omega} ||g||_{L^p(\Omega(x,1))} < +\infty\,. \tag{15}$$

Endowed with such norm, $M^p(\Omega)$ is a Banach space, strictly bigger than the Lebesgue space $L^p(\Omega)$ when Ω is unbounded. Equivalently, we denote by $\tilde{M}^p(\Omega)$ and $M^p_o(\Omega)$ the closure of $L^\infty(\Omega)$ and $C^\infty_o(\Omega)$ in $M^p(\Omega)$, respectively.

Recall that for a function g in $M^p(\Omega)$ the following characterization holds:

- $g \in M^p_\circ(\Omega) \quad \Longleftrightarrow \quad \lim_{\tau\to 0^+}\big(p_g(\tau) + ||(1-\zeta_{1/\tau})g||_{M^p(\Omega)}\big) = 0,$

  where

  $$p_g(\tau) = \sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|E(x,1)|\le\tau}} ||\chi_E g||_{M^p(\Omega)}\,, \quad \tau\in\mathbb{R}_+\,,$$

  and ζ_r, $r\in\mathbb{R}_+$, is a function in $C^\infty_\circ(\mathbb{R}^n)$ such that

  $$0\le\zeta_r\le 1\,, \quad \zeta_{r|B_r} = 1\,, \quad \operatorname{supp}\zeta_r\subset B_{2r}\,.$$

- $g \in \tilde{M}^p(\Omega)$ if and only if the function

  $$\tau_g(t) = \sup_{\substack{E\in\Sigma(\Omega)\\ \sup_{x\in\Omega}|E(x,1)|\le t}} ||\chi_E g||_{M^p(\Omega)} \qquad t\in\mathbb{R}_+\,,$$

  vanishes when t goes to zero.

We want to define the *moduli of continuity* of functions belonging to

$\tilde{M}^{p,\lambda}(\Omega)$ or $M_{\circ}^{p,\lambda}(\Omega)$. To this aim, let us put, for $h \in \mathbb{R}_+$ and $g \in M^{p,\lambda}(\Omega)$,

$$F[g](h) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x\in\Omega} |E(x,1)| \leq \frac{1}{h}}} \| g\, \chi_E \|_{M^{p,\lambda}(\Omega)}.$$

Recall first that for a function $g \in M^{p,\lambda}(\Omega)$ the following characterization holds:

$$g \in \tilde{M}^{p,\lambda}(\Omega) \iff \lim_{h \to +\infty} F[g](h) = 0,$$

while

$$g \in \tilde{M}_{\circ}^{p,\lambda}(\Omega) \iff \lim_{h \to +\infty} \big(F[g](h) + ||(1-\zeta_h) g||_{M^{p,\lambda}(\Omega)} \big) = 0,$$

where ζ_h denotes a function of class $C_o^{\infty}(\mathbb{R}^n)$ such that

$$0 \leq \zeta_h \leq 1\,, \qquad \zeta_{h|_{\overline{B(0,h)}}} = 1\,, \qquad \operatorname{supp} \zeta_h \subset B(0, 2h).$$

Thus, if g is a function in $\tilde{M}^{p,\lambda}(\Omega)$ a *modulus of continuity* of g in $\tilde{M}^{p,\lambda}(\Omega)$ is a map $\widetilde{\sigma}^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) \leq \widetilde{\sigma}^{p,\lambda}[g](h), \quad \lim_{h \to +\infty} \widetilde{\sigma}^{p,\lambda}[g](h) = 0\,.$$

While, if g belongs to $M_o^{p,\lambda}(\Omega)$ a *modulus of continuity* of g in $M_o^{p,\lambda}(\Omega)$

is an application $\sigma_o^{p,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) + \|(1-\zeta_h)\, g\|_{M^{p,\lambda}(\Omega)} \le \sigma_o^{p,\lambda}[g](h),$$

$$\lim_{h\to+\infty} \sigma_o^{p,\lambda}[g](h) = 0\,.$$

Then a *modulus of continuity of* g *in* $\tilde{M}^p(\Omega)$ is a map $\tilde{\sigma}_p[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\tilde{\sigma}_p[g](t) \;\ge\; \tau_g(t) \qquad \forall t \in \mathbb{R}_+\,, \qquad \lim_{t\to0^+} \tilde{\sigma}_p[g](t) = 0\,.$$

Indeed, if $g \in L^p(\Omega)$, the function

$$\omega_p[g](t) := \sup_{\substack{E\in\Sigma(\Omega)\\ |E|\le t}} ||g||_{L^p(E)} \qquad t \in \mathbb{R}_+$$

is clearly non-negative and $\lim_{t\to0^+} \omega_p[g](t) = 0$, so it is a *modulus of continuity* of g in $L^p(\Omega)$.

Finally, we introduce the following functional spaces: if Ω has the property

$$|\Omega(x,r)| \ge A\, r^n \qquad \forall\, x \in \Omega\,, \quad \forall\, r \in\,]0,1] \tag{16}$$

where A is a positive constant independent of x and r, then it is possible to consider the space $BMO(\Omega,\tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\mathrm{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,\tau)} = \sup_{\substack{x\in\Omega\\ r\in]0,\tau]}} ⨍_{\Omega(x,r)} |g - ⨍_{\Omega(x,r)} g| < +\infty\,,$$

where

$$\fint_{\Omega(x,r)} g = |\Omega(x,r)|^{-1} \int_{\Omega(x,r)} g.$$

When $g \in BMO(\Omega) = BMO(\Omega, \tau_A)$, with

$$\tau_A = \sup \left\{ \tau \in \mathbb{R}_+ \ : \ \sup_{\substack{x\in\Omega \\ r\in]0,\tau]}} \frac{r^n}{|\Omega(x,r)|} \leq \frac{1}{A} \right\},$$

we say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega,\tau)} \to 0$ for $\tau \to 0^+$.

Just note that the assumption (16) above implies that Ω is not too 'narrow', and it is clearly satisfied by any domain Ω having the internal cone property, therefore by any $C^{1,1}$-domain.

Let us finish proving an useful lemma:

Lemma 1 *If Ω has the uniform $C^{1,1}$-regularity property and*

$$g, g_x \in \begin{cases} VM^r(\Omega), & r > 2 \text{ for } n = 2, \\ VM^{r,n-r}(\Omega), & r \in]2,n] \text{ for } n > 2, \end{cases}$$

then $g \in VMO(\Omega)$.

PROOF $-$ For $n > 2$ the result can be found in [8], combining Lemma 4.1 and the argument in the proof of Lemma 4.2.

Concerning $n = 2$, we firstly apply a known extension result, see [7] Corollary 2.2, stating that any function g such that $g, g_x \in VM^r(\Omega)$ admits an extension $p(g)$ such that $p(g), (p(g))_x \in VM^r(\mathbb{R}^2)$.

Then, we prove that for all $x_0 \in \mathbb{R}^2$ and $t \in \mathbb{R}_+$, there exists a constant $c \in \mathbb{R}_+$ such that

$$\fint_{B(x_0,t)} \Big| p(g) - \fint_{B(x_0,t)} p(g) \Big| \leq c\big(t^{\frac{r-2}{r}} \|(p(g))_x\|_{L^r(B(x_0,t))}\big), \tag{17}$$

indeed, in view of the above considerations, if (17) holds true, one has that $p(g) \in VMO(\mathbb{R}^2)$, so $g \in VMO(\Omega)$.

Consider the function

$$g^* : z \in \mathbb{R}^2 \to p(g)(x_0 + tz) \in \mathbb{R}.$$

By Poincaré-Wirtinger inequality and Hölder inequality one gets

$$\fint_{B(x_0,t)} |p(g)(x) - \fint_{B(x_0,t)} p(g)(x)| =$$

$$\pi^{-1} \int_{B(0,1)} |g^*(z) - \fint_{B(0,1)} g^*(z)| \leq c_1 \int_{B(0,1)} |(g^*)_z(z)| =$$

$$c_1\, t^{-1} \int_{B(x_0,t)} |(p(g))_x(x)| \leq c_1\, t^{-1} |B(x_0,t)|^{\frac{r-1}{r}} \|(p(g)_x\|_{L^r(B(x_0,t))},$$

this gives (17). □

A more detailed account of properties of the above defined function spaces can be found in [25, 43, 50, 52, 53].

Chapter 1

Weight functions and weighted Sobolev spaces

The general framework in which we develop our work is the relationship between the Dirichlet problem associated to a linear elliptic operator and the Sobolev spaces in which its solution may live.

The main goal of this chapter is to introduce the weight functions and their corresponding weighted Sobolev spaces to investigate about some reasons that lead to choose certain weight functions. Finally, two new classes of weighted functions are studied.

1.1 Why the weighted Sobolev spaces?

Let us start with basic definitions.

Definition 1.1.1 *Let Ω be an open subset in $\mathbb{R}^n$. By the symbol $\mathbb{T}(\Omega)$,*

we denote the set of all measurable almost everywhere (a.e.) in Ω, *positive and finite functions* $t = t(x)$, $x \in \Omega$.
Elements of $\mathbb{T}(\Omega)$ *will be called* **weight functions**.

Definition 1.1.2 *Let* $\Omega \subset \mathbb{R}^n$, $p \geq 1$, $t \in \mathbb{T}(\Omega)$. *By the symbol* $L_t^p(\Omega)$ *we denote the set of all measurable functions* $u = u(x)$, $x \in \Omega$ *such that*

$$||u||^p_{L_t^p(\Omega)} = \int_\Omega |u(x)|^p t(x) dx < +\infty$$

For $t(x) \equiv 1$ *we obtain the usual Lebesgue space* $L^p(\Omega)$.

Remark 1.1.3 $L_t^p(\Omega)$ *equipped with the norm* $|| \cdot ||_{L_t^p(\Omega)}$ *is a Banach space.*

Definition 1.1.4 *Let* $\Omega \subset \mathbb{R}^n$ *a domain with a boundary* $\partial\Omega$, t *a vector of non-negative (positive a.e.) measurable functions on* Ω, *i.e. a weight*

$$t = \{t_\alpha = t_\alpha(x), x \in \Omega, |\alpha| \leq k\}$$

where k *is a non-negative integer,* α *is a multiindex, i.e.,* $\alpha \in \mathbb{N}_0^n$ *or equivalently*

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \qquad \alpha_i \in \mathbb{N}_0$$

$$|\alpha| = \alpha_1 + \alpha_2 + ...\alpha_n.$$

Let us define the Sobolev space with weight t, $W_t^{k,p}(\Omega)$, *where* p *is a number,* $1 \leq p \leq +\infty$, *as the set of all functions* $u \in L_t^p(\Omega) \cap L^1_{loc}(\Omega)$

such that their distributional derivatives $\partial^\alpha u$*,* $\forall |\alpha| \leq k$ *are again elements of* $L^p_t(\Omega) \cap L^1_{loc}(\Omega)$ *(i.e.,* $\partial^\alpha u$ *are regular distributions).*
The expression

$$||u||_{W^{k,p}_t(\Omega)} = \left(\sum_{|\alpha| \leq k} ||\partial^\alpha u||_{L^p_t(\Omega)} \right)^{\frac{1}{p}} \tag{1.1}$$

obviously is a norm on the linear space $W^{k,p}_t(\Omega)$*.*

The usefulness of the spaces $L^p_t(\Omega)$ is self-evident, for example, in the theory of orthogonal polynomials. Concerning the weighted Sobolev space $W^{k,p}_t(\Omega)$, as a remarkable example, we refer to the application of these spaces in the theory of boundary-value problems for PDEs.

Let us start to investigate the homogeneous Dirichlet problem associated to a Laplace operator:

$$\begin{cases} -\Delta u + u = f \\ u_{|\partial\Omega} = 0. \end{cases} \tag{1.2}$$

As everyone knows, after multiplying the equation by the function u, integrating the resulting identity over Ω and using the Green's Formula, we obtain - thanks to the boundary condition - the integral identity

$$\sum_{i=1}^n \int_\Omega \left(\frac{\partial u}{\partial x_i} \right)^2 dx + \int_\Omega u^2 dx = \int_\Omega f u dx.$$

The left hand side of this identity represents the square of the norm of the function u in the Sobolev space $W^{1,2}(\Omega)$, so that the relation can be written also in the form

$$||u||^2_{W^{1,2}(\Omega)} = \int_\Omega fu dx.$$

This relation is the starting point of the theory of the weak solutions of boundary-value problem for elliptic equations.

Let us consider, now, a linear elliptic differential operator L of the second order (for simplicity)

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a. \tag{1.3}$$

We shall assign a bilinear form

$$a(u, v)$$

defined for u, v from a certain subspace $V \subset W^{1,2}(\Omega)$ (the subspace V being determined by the boundary conditions), and instead of solving the boundary-value problem for the equation

$$Lu = f$$

we consider the identity

$$a(u,v) =< f,v > \ \forall\, v \in V. \tag{1.4}$$

The equivalence below is essential for the existence of a solution of the problem (1.4)

$$a(u,u) = ||u||^2_{W^{1,2}(\Omega)}. \tag{1.5}$$

The possibility to resolve this equation depends on the existence of a space to which the function u belongs. In several situations, it's not possible to find this function in the classical Sobolev spaces but it's necessary to modify suitably the spaces in order to obtain this function.

Let us investigate some of these situations:

- **Equations with perturbed ellipticity**: instead of the equation (1.2), we'll concern a different equation

  $$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\rho_i(x)\frac{\partial u}{\partial x_i}\right) + \rho_0(x)u = f \ \ on \ \ \Omega$$

  where the coefficients of the operator $\rho_i = \rho_i(x), i = 0,...,N,$ are non-negative functions defined on Ω,

  – degenerate: $\rho_i(x) \to 0$ for $x \to x_0 \in \partial\Omega$.

  or

  – have a singularity: $\rho_i(x) \to \infty$ for $x \to x_0 \in \partial\Omega$.

With the same procedure as the problem (1.2), we arrive at the integral identity

$$\sum_{i=1}^{n}\int_{\Omega}\left(\frac{\partial u}{\partial x_i}\right)^2\rho_i(x)dx+\int_{\Omega}u^2\rho_0(x)dx=\int_{\Omega}fudx.$$

Consequently, if L is a linear differential operator with perturbed ellipticity, then we can still associated it with the corresponding bilinear form $a(u,v)$. Indeed, if there is a suitable weight t such that

$$a(u,u)=||u||^2_{W_t^{1,2}(\Omega)} \tag{1.6}$$

we can try to solve the problem (1.4); obviously in this case $V\subset W_t^{1,2}(\Omega)$.

So, the weighted spaces make possible to enlarge the class of equations which are solvable by functional-analytical method.

- **Nonhomogeneous Dirichlet problem**: To solve the boundary value problem:

$$\begin{cases}-\Delta u+u=f\\ \\ u_{|\partial\Omega}=g;\end{cases}$$

in the classical Sobolev space, we have to satisfy two conditions:

1. $g\in W^{\frac{1}{2},2}(\partial\Omega)$, i.e. g is the trace of $\tilde{g}\in W^{1,2}(\Omega)$ on $\partial\Omega$,
2. f is a continuous linear functional over the space $\overset{\circ}{W}{}^{1,2}(\Omega)$, i.e. $f\in W^{-1,2}(\Omega)$.

If one of these conditions fails the classical theory of Sobolev spaces cannot be applied. We can make an attempt to find a suitable weight t for which the theory of weak solutions can be extended also to the case of the weighted space $W_t^{1,2}(\Omega)$. Indeed, we look for certain weights t for which there exists analogue of the known existence and uniqueness theorem for the weak solution of the classical boundary value problem. Otherwise, contrary to the previous case, the weight it's not a priori given by the equation.

- **Unbounded domains**: In this case, in addition to the boundary condition on $\partial\Omega$ required by the Dirichlet problem, we need to ask also conditions at infinity which prescribes the behaviour of the solutions $u(x)$ for $|x| \to \infty$. These requirements can be described throught weight functions. So, the weighted spaces allow to study also functions defined on unbounded domains. Main results about the above application are due to **L.D.Kundjavcev** and his successors **B.Hanouzet**, **A.Avantaggiati**, **M.Troisi** and **R.A.Adams**.

- **A domain with corners or edges**: The reflection of these geometric features of the domain Ω may be found in the properties of solution of boundary value problems on Ω. Near of a corner or an edge the solution u of the boundary value problem may have a singularity well characterized by a suitable weight. This weight is most usually a power of the distance from the singular set on $\partial\Omega$.

So, a weighted space can help us to describe the qualitative properties of solutions of boundary value problems. On the other hand, it may have a "*practical*" aspect as well: weighted spaces have proved useful, for example, in connection with the approximate solution of boundary value problems by means the finite element method.

1.2 How to choose suitably a weight

The most reasonably motivation to choose a class of weight functions than another one lies in looking for those classes for which the corresponding weighted Sobolev space is guaranteed to be complete, i.e. a Banach space. Further, it is shown how to modify the definition of the weighted space if the weight function do not belong to the class mentioned.

Definition 1.2.1 *Let $p > 1$. We shall say that a weight function $t \in \mathbb{T}(\Omega)$ satisfies condition $B_p(\Omega)$ and write $t \in B_p(\Omega)$ if*

$$t^{-\frac{1}{(p-1)}} \in L^1_{loc}(\Omega).$$

Theorem 1.2.2 *Let $\Omega \subset \mathbb{R}^n$ be an open set, $p > 1$, $t \in B_p(\Omega)$. Then*

$$L^p_t(\Omega) \hookrightarrow L^1_{loc}(\Omega)$$

($\hookrightarrow$ continous embedding).

Using the usual assumption of a regular distribution in $\mathfrak{D}'(\Omega)$ of a function in $L^1_{loc}(\Omega)$, we conclude that

$$L^p_t(\Omega) \subset L^1_{loc}(\Omega) \subset \mathfrak{D}'(\Omega) \tag{1.7}$$

for $t \in B_p(\Omega)$. Therefore, for functions $u \in L^p_t(\Omega)$ with $t \in B_p(\Omega)$, the distributional derivatives $\partial^\alpha u$ of u have sense.

Remark 1.2.3 *If the weight function t satisfies the condition $B_p(\Omega)$, in view of (1.7), the assumption $\partial^\alpha u \in L^p_t(\Omega) \cap L^1_{loc}(\Omega)$ in the definition (1.1.4) can be replaced by the assumption $\partial^\alpha u \in L^p_t(\Omega)$.*

Theorem 1.2.4 *If $t \in B_p(\Omega)$, the space $W^{1,p}_t(\Omega)$ is a Banach space if equipped with the norm (1.1).*

Now, we introduce exceptional sets definition of the weighted Sobolev spaces which causes the non-completeness. These sets are composed by the points on that the weight functions are not $B_p(\Omega)$.

Definition 1.2.5 *Let $t \in \mathbb{T}(\Omega)$, $p > 1$ and denote*

$$M_p(t) = \{x \in \Omega : \int_{\Omega \cap U(x)} t^{-\frac{1}{p-1}}(y)dy = +\infty \; \forall \; U(x) \; of \; x\}$$

Obviously, $M_p(t) = \emptyset$ for $t \in B_p(\Omega)$.

Let us denote

$$\mathbb{B} = \bigcup_{t \notin B_p(\Omega)} M_p(t) \tag{1.8}$$

Definition 1.2.6 *Let Ω, p and t be as in definition (1.1.1), with $t \in \mathbb{T}(\Omega)$. Let $\mathbb{B}$ be the set from (1.8). Then we define the Sobolev space with weight t,*

$$W_t^{1,p}(\Omega)$$

as the space $W_t^{1,p}(\Omega \backslash \mathbb{B})$, considered in the sense of definition (1.1.4)

Remark 1.2.7 *Another way how to guarantee the completeness of the weighted Sobolev space is to define it as the completion of the set $W_t^{1,p}(\Omega)$ from definition (1.1.4) with respect to the norm (1.1). However, in this case the completion could contain nonregular distributions or functions whose distributional derivatives are not regular distributions.*
Therefore, definition (1.2.6) seems to be more natural.

Let us introduce two new classes of weight functions. Obviously, the related weighted Sobolev spaces are Banach spaces. We work with weight functions or s-th power of them. Their role is to check the run of the functions, and their derivatives, belonging to weighted Sobolev spaces. Specifically, the weight functions fix the behaviour of those functions at infinity on unbounded domains and correct it near not regular parts of the boundary of the domain.

1.3 $\mathcal{C}^k(\overline{\Omega})$ - weight functions

Let Ω be an open subset of $\mathbb{R}^n$, not necessarily bounded, $n \geq 2$. We introduce a class of weight functions defined on $\overline{\Omega}$. To this aim, given

$k \in \mathbb{N}_0$, we consider a function $\rho : \bar{\Omega} \to \mathbb{R}_+$ such that $\rho \in C^k(\bar{\Omega})$ and

$$\sup_{x \in \Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty, \quad \forall\, |\alpha| \leq k. \tag{1.9}$$

Remark 1.3.1 *If $\rho \in C^k(\bar{\Omega})$ and satisfies (1.9), then $\rho, \rho^{-1} \in L^\infty_{loc}(\bar{\Omega})$.*

As an example, we can think of the function

$$\rho(x) = (1 + |x|^2)^t, \quad t \in \mathbb{R}.$$

In the following lemma, we show a property, needed in the sequel, concerning this class of weight functions.

Lemma 1.3.2 *If assumption* (1.9) *is satisfied, then*

$$\sup_{x \in \Omega} \frac{|\partial^\alpha \rho^s(x)|}{\rho^s(x)} < +\infty \quad \forall s \in \mathbb{R}, \quad \forall\, |\alpha| \leq k. \tag{1.10}$$

Proof $-$ The proof is obtained by induction. From (1.9) we get

$$|(\rho^s)_{x_i}| = |s \rho^{s-1} \rho_{x_i}| \leq c_1 \rho \rho^{s-1} = c_1 \rho^s, \quad i = 1, ..., n,$$

with c_1 positive constant depending only on s. Thus (1.10) holds for $|\,\alpha| = 1$.

Now, let us assume that (1.10) holds for any β such that $|\beta| < |\alpha|$ and any $s \in \mathbb{R}$, and fix a β such that $|\beta| = |\alpha| - 1$. Then, using (1.9) and by the induction hypothesis written for $s-1$, we have

$$|\partial^\alpha \rho^s| = |\partial^\beta (\rho^s)_{x_i}| = |\partial^\beta (s\rho^{s-1}\rho_{x_i})| \leq$$
$$c_2 \sum_{\gamma\leq\beta} |\partial^{\beta-\gamma}\rho_{x_i} \ \partial^\gamma \rho^{s-1}| \leq c_3 \rho\rho^{s-1} = c_3\rho^s, \text{ for } i=1,...,n,$$

with c_3 positive constant depending only on s. Hence, (1.10) holds true also for α. □

Now, let us study some properties of the class of weighted Sobolev spaces with weight function of the above mentioned type.

We can define for $k \in \mathbb{N}_0$, $p \in [1,+\infty[$ and $s \in \mathbb{R}$, given a weight function ρ satisfying (1.9), the space $W^{k,p}_s(\Omega)$ of distributions u on Ω such that $\rho^s \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$, equipped with the norm:

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha|\leq k} \|\rho^s \partial^\alpha u\|_{L^p(\Omega)} < +\infty, \tag{1.11}$$

and we denote by $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ the closure of $C^\infty_\circ(\Omega)$ in $W^{k,p}_s(\Omega)$ and put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$.

Lemma 1.3.3 *Let $k \in \mathbb{N}_0$, $p \in [1,+\infty[$ and $s \in \mathbb{R}$. If assumption* (1.9) *is satisfied, then there exist two constants $c_1, c_2 \in \mathbb{R}_+$ such that*

$$c_1||u||_{W^{k,p}_s(\Omega)} \leq ||\rho^t u||_{W^{k,p}_{s-t}(\Omega)} \leq c_2||u||_{W^{k,p}_s(\Omega)}, \tag{1.12}$$

$\forall t \in \mathbb{R}$, $\forall u \in W^{k,p}_s(\Omega)$, with $c_1 = c_1(t)$ and $c_2 = c_2(t)$.

Proof − Observe that from (1.10) we have

$$|\partial^\alpha(\rho^t u)| \le c_1 \sum_{\beta\le\alpha} |\partial^{\alpha-\beta}\rho^t\ \ \partial^\beta u| \le c_2|\rho^t\partial^\beta u|,$$

with $c_2 \in \mathbb{R}_+$ depending only on t. This entails the inequality on the right hand side of (1.12).

To get the left hand side inequality, it is enough to show that

$$|\rho^t\partial^\alpha u| \le c_3 \sum_{\beta\le\alpha} |\partial^\beta(\rho^t u)|, \tag{1.13}$$

with $c_3 \in \mathbb{R}_+$ depending only on t.

We will prove (1.13) by induction. From (1.10) one has

$$|\rho^t u_{x_i}| = |(\rho^t u)_{x_i} - (\rho^t)_{x_i} u| \le c_4\big((\rho^t u)_x + \rho^t|u|\big),$$

for $i = 1, ..., n$, with $c_4 \in \mathbb{R}_+$ depending only on t. Hence, (1.13) holds for $|\alpha| = 1$.

If (1.10) holds for any β such that $|\beta| < |\alpha|$, then, using again (1.10) and by the induction hypothesis, we have

$$|\rho^t\partial^\alpha u| \le |\partial^\alpha(\rho^t u)| + c_5 \sum_{\beta<\alpha} |\partial^{\alpha-\beta}\rho^t||\partial^\beta u| \le$$
$$|\partial^\alpha(\rho^t u)| + c_6 \sum_{\beta<\alpha} |\rho^t\partial^\beta u| \le c_7 \sum_{\beta\le\alpha} |\partial^\beta(\rho^t u)|,$$

with $c_7 \in \mathbb{R}_+$ depending only on t. □

Let us specify a general density result, true whenever the Sobolev space is weighted with a weight function in the class $L^\infty_{loc}(\overline{\Omega})$ with its inverse.

Lemma 1.3.4 *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$. If Ω has the segment property and assumption* (1.9) *is satisfied, then $\mathcal{D}(\bar{\Omega})$ is dense in $W^{k,p}_s(\Omega)$.*

Proof − The proof follows by Lemma 2.2 in [46], since clearly both $\rho, \rho^{-1} \in L^\infty_{loc}(\bar{\Omega})$. $\square$

This allows us to prove the following inclusion:

Lemma 1.3.5 *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$. If Ω has the segment property and assumption* (1.9) *is satisfied, then*

$$W^{k,p}_s(\Omega) \cap \overset{\circ}{W}{}^{k,p}(\Omega) \subset \overset{\circ}{W}{}^{k,p}_s(\Omega).$$

Proof − The density result stated in Lemma 1.3.4 being true, we can argue as in the proof of Lemma 2.1 of [22] to obtain the claimed inclusion. $\square$

From this last lemma we easily deduce that, if Ω has the segment property, also $C^k_o(\Omega) \subset \overset{\circ}{W}{}^{k,p}_s(\Omega)$.

Now, we introduce the essential property of $\mathcal{C}^k(\overline{\Omega})$-weight class, named topological isomorphism.

Lemma 1.3.6 *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$. If Ω has the segment* *property and assumption* (1.9) *is satisfied, then the map*

$$u \longrightarrow \rho^s u$$

defines a topological isomorphism from $W^{k,p}_s(\Omega)$ *to* $W^{k,p}(\Omega)$ *and from* $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ *to* $\overset{\circ}{W}{}^{k,p}(\Omega)$.

Proof − The first part of the proof easily follows from Lemma 1.3.3 with $t = s$. Let us show that $u \in \overset{\circ}{W}{}^{k,p}_s(\Omega)$ if and only if $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$.

If $u \in \overset{\circ}{W}{}^{k,p}_s(\Omega)$, there exists a sequence $(\phi_h)_{h\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to u in $W^{k,p}_s(\Omega)$. Therefore, fixed $\varepsilon \in \mathbb{R}_+$, there exists $h_0 \in \mathbb{N}$ such that

$$\|\rho^s(\phi_h - u)\|_{W^{k,p}(\Omega} < \frac{\varepsilon}{2}, \quad \forall h > h_0. \tag{1.14}$$

Fix $h_1 > h_0$, clearly $\rho^s\phi_{h_1} \in \overset{\circ}{W}{}^{k,p}(\Omega)$, because of its compact support. Therefore, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to $\rho^s\phi_{h_1}$ in $W^{k,p}(\Omega)$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\|\psi_n - \rho^s\phi_{h_1}\|_{W^{k,p}(\Omega)} < \frac{\varepsilon}{2}, \quad \forall n > n_0. \tag{1.15}$$

Putting together (1.14) and (1.15) we get

$$\|\psi_n - \rho^s u\|_{W^{k,p}(\Omega)} \le \|\psi_n - \rho^s\phi_{h_1}\|_{W^{k,p}(\Omega)} + \|\rho^s\phi_{h_1} - \rho^s u\|_{W^{k,p}(\Omega)} < \varepsilon,$$

$\forall\, n > n_0$. Thus $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$. Viceversa, if we assume that $\rho^s u \in \overset{\circ}{W}{}^{k,p}(\Omega)$, we find a sequence $(\phi_h)_{h\in\mathbb{N}} \subset C^\infty_o(\Omega)$ converging to $\rho^s u$ in

$W^{k,p}(\Omega)$. Hence, there exists $h_0 \in \mathbb{N}$ such that

$$\|\rho^{-s}\phi_h - u\|_{W_s^{k,p}(\Omega)} < \frac{\varepsilon}{2}, \quad \forall h > h_0. \tag{1.16}$$

Fix $h_1 > h_0$, since $\rho^{-s}\phi_{h_1} \in C_o^k(\Omega)$, which is contained in $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ by Lemma 1.3.5, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset C_o^\infty(\Omega)$ converging to $\rho^{-s}\phi_{h_1}$ in $\overset{\circ}{W}{}_s^{k,p}(\Omega)$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\|\psi_n - \rho^{-s}\phi_{h_1}\|_{W_s^{k,p}(\Omega)} < \frac{\varepsilon}{2}, \quad \forall n > n_0. \tag{1.17}$$

From (1.16) and (1.17) we get

$$\|\psi_n - u\|_{W_s^{k,p}(\Omega)} \leq \|\psi_n - \rho^{-s}\phi_{h_1}\|_{W_s^{k,p}(\Omega)} + \|\rho^{-s}\phi_{h_1} - u\|_{W_s^{k,p}(\Omega)} < \varepsilon,$$

$\forall\, n > n_0$. So that $u \in \overset{\circ}{W}{}_s^{k,p}(\Omega)$. □

1.4 $\mathcal{G}(\Omega)$ - weight functions

Here, we introduce a class of weight functions defined on Ω, an open subset of $\mathbb{R}^n$, not necessarily bounded, with $n \geq 2$, and $d \in \mathbb{R}_+$. Denoted by $G_d(\Omega)$ the set of all measurable functions $m : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<d}} \frac{m(x)}{m(y)} < +\infty \tag{1.18}$$

we say $\mathcal{G}(\Omega)$ be the class of weight functions defined as:

$$\mathcal{G}(\Omega) = \bigcup_{d \in \mathbb{R}_+} G_d(\Omega).$$

Examples of functions in $\mathcal{G}(\Omega)$ are functions of distance type, as:

$$m(x) = e^{t|x|}\,, \quad m(x) = (1 + |x|^2)^t\,, \quad x \in \Omega,\, t \in \mathbb{R}.$$

In order to pick out $\mathcal{G}(\Omega)$ functions we draw up a list of their properties:

- $m \in \mathcal{G}(\Omega)$ if and only if there exist $d, \gamma \in \mathbb{R}_+$ such that

$$\gamma^{-1}\, m(y) \le m(x) \le \gamma\, m(y) \qquad \forall\, y \in \Omega\,, \quad \forall\, x \in \Omega(y,d) \tag{1.19}$$

  where $\gamma \in \mathbb{R}_+$ is independent of x and y.

- if $m \in \mathcal{G}(\Omega)$ then

$$m\,,\ m^{-1}\ \in L^\infty_{\mathrm{loc}}(\bar{\Omega}). \tag{1.20}$$

- $m \in \mathcal{G}(\Omega)$ if and only if $\exists\, d \in \mathbb{R}_+$ such that

$$\sup_{\substack{x,y \in \Omega \\ |x-y|<d}} \left| \log \frac{m(x)}{m(y)} \right| < +\infty. \tag{1.21}$$

- if $m \in \mathcal{G}(\Omega)$, then:

$$m^s \in \mathcal{G}(\Omega), \quad \lambda m \in \mathcal{G}(\Omega) \qquad \forall s \in \mathbb{R}, \lambda \in \mathbb{R}_+.$$

- **Lemma 1.4.1** *Let m be a positive function defined on Ω. If* $\log m \in \mathrm{Lip}(\Omega)$ *then $m \in \mathcal{G}(\Omega)$.*

  PROOF − By the hypothesis, there exists a constant $L \in \mathbb{R}_+$ such that for each $x, y \in \Omega$

$$|logm(x) - logm(y)| \leq L|x - y|. \tag{1.22}$$

  Let $x, y \in \Omega$ such that $|x-y| < d$ ($d \in \mathbb{R}_+$). From (1.22) we deduce that

$$\left|log\frac{m(x)}{m(y)}\right| \leq Ld \qquad \forall y \in \Omega, \quad \forall x \in \Omega(y, d)$$

  and we have the result. □

- **Lemma 1.4.2** *(REGULARIZATION FUNCTION σ)*
  If $m \in \mathcal{G}(\Omega)$ and Ω has the cone property, then there exists a function $\sigma \in \mathcal{G}(\Omega) \cap C^\infty(\bar{\Omega})$ such that

$$c_1 m(x) \leq \sigma(x) \leq c_2 m(x) \qquad \forall x \in \Omega, \tag{1.23}$$

$$\sup_{x\in\Omega} \frac{|\partial^\alpha \sigma(x)|}{\sigma(x)} < +\infty \qquad \forall \alpha \in \mathbb{N}_0^n, \tag{1.24}$$

$$\sup_{x\in\Omega} \frac{|\partial^\alpha \sigma^s(x)|}{\sigma^s(x)} < +\infty \qquad \forall \alpha \in \mathbb{N}_0^n,\ \forall s \in \mathbb{R} \tag{1.25}$$

  where $c_1, c_2 \in \mathbb{R}_+$ are dependent only on n, Ω, m.

  PROOF − Since $m \in \mathcal{G}(\Omega)$ there exists a positive number d such that $m \in \mathcal{G}_d(\Omega)$. We assign a function $g \in C_\circ^\infty(\mathbb{R}^n)$ such that

$$g \geq 0\,, \quad g_{|B_{\frac{1}{2}}} = 1\,, \quad \operatorname{supp} g \subset B_1$$

and put

$$\sigma : \, x \in \Omega \longrightarrow \int_\Omega m(y)\, g\Big(\frac{x-y}{d}\Big)\, dy\,.$$

Since

$$\sigma(x) = \int_{\Omega(x,d)} m(y)\, g\Big(\frac{x-y}{d}\Big)\, dy \qquad \forall x \in \Omega\,,$$

by (1.19), it follows that: $\forall\, x \in \Omega, \forall\, y \in \Omega(x,d)$

$$c^{-1}m(x) \int_{\Omega(x,d)} g\Big(\frac{x-y}{d}\Big)\, dy \leq \int_{\Omega(x,d)} m(y)\, g\Big(\frac{x-y}{d}\Big)\, dy \leq$$

$$\leq c\, m(x) \int_{\Omega(x,d)} g\Big(\frac{x-y}{d}\Big)\, dy.$$

So, on the one hand

$$c\, m(x) \int_{\Omega(x,d)} g\Big(\frac{x-y}{d}\Big)\, dy \leq c\, m(x) \big(\sup_{\Omega(x,d)} g\big) |\Omega(x,d)| \leq$$

$$\leq c\, m(x)\, \overline{c}\, \omega_n d^n = c_2(n, m, \Omega)\, m(x),$$

on the other hand, by hypotheses on function g:

$$c_1(n, m, \Omega)\, m(x) \leq c^{-1} m(x) \overline{\overline{c}}\, \omega_n \left(\frac{d}{2}\right)^n \leq c^{-1} m(x) \int_{\Omega(x,\frac{d}{2})} g\Big(\frac{x-y}{d}\Big)\, dy \leq$$

$$\leq \int_{\Omega(x,d)} m(y)\, g\Big(\frac{x-y}{d}\Big)\, dy = \sigma(x)$$

then, putting together the previous estimates we obtain the (1.23). Thus, by the equivalence (1.23) and continuity of g, $\sigma \in \mathcal{G}(\Omega) \cap C^\infty(\bar{\Omega})$.

Moreover, using jet (1.19), for all $\alpha \in \mathbb{N}_0^n$ and $x \in \Omega$, we have:

$$|\partial^\alpha \sigma(x)| \leq \gamma\, m(x) d^{-|\alpha|} \int_{\Omega(x,d)} \Big| g^{(|\alpha|)}\Big(\frac{x-y}{d}\Big)\Big|\, dy \leq c_3 m(x),$$

where c_3 depends on n, Ω, m, α, and then (1.24) follows.

By the induction procedure on the length of $\alpha \in \mathbb{N}_0^n$, it is easy to prove (1.25).

- **Lemma 1.4.3** *If Ω has the property that there exist $r_0 \in \mathbb{R}_+$ and $x_0 \in \Omega \backslash B_{r_0}$ such that $\overline{xx_0} \subset \Omega \ \ \forall x \in \Omega\backslash B_{r_0}$, then for any $m \in \mathcal{G}(\Omega)$ we have*

$$c_0^{-1} e^{-c|x|} \leq m(x) \leq c_0 e^{c|x|} \qquad \forall x \in \Omega,$$

where c and c_0 depend only on n, Ω and m.

Proof $-$ Fix $x \in \Omega$. If $x \in \Omega\backslash B_{r_0}$ then $\overline{xx_0} \subset \Omega$ and by Lagrange's theorem, using (1.24), we have

$$|log\sigma(x) - log\sigma(x_0)| = \sum_{i=1}^n \frac{\sigma_{x_i}(x)}{\sigma(x)} \cdot |x - x_0| \leq c|x - x_0| \qquad (1.26)$$

where $c \in \mathbb{R}_+$ depends on n, Ω, m. So, with easy computations and from (1.12), we have the result.

Otherwise if $x \in \Omega \cap B_{r_0}$, from (1.20), we have the result. $\square$

If $m \in \mathcal{G}(\Omega)$, $k \in \mathbb{N}_0$, $1 \leq p < +\infty$ and $s \in \mathbb{R}$, we define the space $W^{k,p}_s(\Omega)$ of distributions u on Ω such that $m^s \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$, equipped with the norm

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha| \leq k} \|m^s \partial^\alpha u\|_{L^p(\Omega)}. \tag{1.27}$$

Moreover, denote by $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ the closure of $C^\infty_\circ(\Omega)$ in $W^{k,p}_s(\Omega)$ and put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$.

A more detailed account of properties of the above defined spaces can be found, for instance, in [54]. Now, by (1.25), we can easily deduce the following topological map. It allows to pass from weighted Sobolev spaces to classical Sobolev spaces in order to take advantage of their theory.

Lemma 1.4.4 *Let $k \in \mathbb{N}_0$, $1 \leq p < +\infty$ and $s \in \mathbb{R}$. If Ω has the cone property, $m \in \mathcal{G}(\Omega)$ and σ is the function defined in Lemma 1.4.2, then the map*

$$u \longrightarrow \sigma^s u$$

defines a topological isomorphism from $W^{k,p}_s(\Omega)$ to $W^{k,p}(\Omega)$ and from $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ to $\overset{\circ}{W}{}^{k,p}(\Omega)$.

We can obtain the above equivalence as for $\mathcal{C}^k(\overline{\Omega})$ weight functions, here we underline only that for topological isomorphism from $W^{k,p}_s(\Omega)$ to $W^{k,p}(\Omega)$ (or from $\mathring{W}^{k,p}_s(\Omega)$ to $\mathring{W}^{k,p}(\Omega)$) we means

$$u \in W^{k,p}_s(\Omega) \Leftrightarrow \sigma^s u \in W^{k,p}(\Omega)$$

or equivalently that $\exists\, c_1, c_2 \in \mathbb{R}_+$ (independent of u) such that

$$c_1 ||\sigma^s u||_{W^{k,p}} \leq ||u||_{W^{k,p}_s} \leq c_2 ||\sigma^s u||_{W^{k,p}(\Omega)},$$

1.5 Some embedding results in $\mathcal{G}(\Omega)$ - weighted Sobolev spaces

In the study of several elliptic problems with solutions in Sobolev spaces (with or without weight), at the aim to obtain existence and uniqueness theorems it is sometimes necessary to estabilish regularity results and a priori estimates for the solutions. These issues rely on some embeddings for the operator

$$u \in W^{k,p}_s(\Omega) \to gu \in L^p_s(\Omega).$$

Moreover, if L is the associated operator to the corresponding elliptic problem, these results can prove the boundedness and the compactness of L, when g is a coefficient of the operator.

Let m be a function of class $\mathcal{G}(\Omega)$. We consider the following condition:

(h_0) Ω has the cone property, $p \in]1, +\infty[, s \in \mathbb{R}, k, t$ are numbers such that:

$$k \in \mathbb{N}, \ \ t \geq p, \ t \geq \frac{n}{k}, \ t > p \ \text{ if } \ p = \frac{n}{k}, \ g \in M^t(\Omega).$$

By Theorem 3.1 of [24] we easily obtain the following.

Theorem 1.5.1 *If the assumption* (h_0) *holds, then for any* $u \in W^{k,p}_s(\Omega)$ *we have* $gu \in L^p_s(\Omega)$ *and*

$$||gu||_{L^p_s(\Omega)} \leq c\,||g||_{M^t(\Omega)}||u||_{W^{k,p}_s(\Omega)}, \tag{1.28}$$

with c *dependent only on* Ω, n, k, p *and* t*.*

Corollary 1.5.2 *If the assumption* (h_0) *holds and* $g \in \tilde{M}^t(\Omega)$*, then for any* $\varepsilon \in \mathbb{R}_+$ *there exists a constant* $c(\varepsilon) \in \mathbb{R}_+$ *such that*

$$||gu||_{L^p_s(\Omega)} \leq \varepsilon ||u||_{W^{k,p}_s(\Omega)} + c(\varepsilon)||u||_{L^p_s(\Omega)} \ \ \forall u \in W^{k,p}_s(\Omega), \tag{1.29}$$

where $c(\varepsilon)$ *depends only on* $\varepsilon, \Omega, n, k, p, t, \tilde{\sigma}[g]$*.*

PROOF − Fix $\varepsilon > 0$ and let c be the constant in (1.28). Since $g \in \tilde{M}^t(\Omega)$, then there exists $g_\varepsilon \in L^\infty(\Omega)$ such that $||g - g_\varepsilon||_{M^t(\Omega)} < \frac{\varepsilon}{c}$. By Theorem 1.5.1

$$||gu||_{L^p_s(\Omega)} \leq c\,||g - g_\varepsilon||_{M^t(\Omega)}||u||_{W^{k,p}_s(\Omega)} + ||g_\varepsilon||_{L^\infty(\Omega)}||u||_{L^p_s(\Omega)}$$

for any u in $W^{k,p}_s(\Omega)$, and then the result follows. □

Corollary 1.5.3 *If the assumption (h_0) holds and $g \in M^t_\circ(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ there exist a constant $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset\subset \Omega$ with the cone property such that*

$$||gu||_{L^p_s(\Omega)} \leq \varepsilon||u||_{W^{k,p}_s(\Omega)} + c(\varepsilon)||u||_{L^p(\Omega_\varepsilon)} \quad \forall u \in W^{k,p}_s(\Omega), \tag{1.30}$$

where $c(\varepsilon)$ and Ω_ε depend only on $\varepsilon, \Omega, n, k, p, m, s, t, \sigma_\circ[g]$.

PROOF − Fix $\varepsilon > 0$ and let c be the constant in (1.28). Since $g \in M^t_\circ(\Omega)$, there exists $g_\varepsilon \in C^\infty_\circ(\Omega)$ such that $||g - g_\varepsilon||_{M^t(\Omega)} < \frac{\varepsilon}{c}$. Let Ω_ε be a bounded open subset of Ω, with the cone property, such that supp $g_\varepsilon \subset \Omega_\varepsilon$, hence by Theorem 1.5.1 and (1.20), it follows that

$$\begin{aligned} ||gu||_{L^p_s(\Omega)} &\leq c\,||g - g_\varepsilon||_{M^t(\Omega)}||u||_{W^{k,p}_s(\Omega)} + ||g_\varepsilon u||_{L^p_s(\Omega_\varepsilon)} \\ &\leq \varepsilon||u||_{W^{k,p}_s(\Omega)} + ||g_\varepsilon m^s||_{L^\infty(\Omega_\varepsilon)}||u||_{L^p(\Omega_\varepsilon)} \qquad (1.31) \end{aligned}$$

for any u in $W^{k,p}_s(\Omega)$, and then we have the result. □

Theorem 1.5.4 *If the assumption* (h_0) *holds and* $g \in M_\circ^t(\Omega)$*, then the operator*

$$u \in W_s^{k,p}(\Omega) \longrightarrow gu \in L_s^p(\Omega) \tag{1.32}$$

is compact.

PROOF – Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions which weakly converges to zero in $W_s^{k,p}(\Omega)$. Therefore there exists $b \in \mathbb{R}_+$ such that $||u_n||_{W_s^{k,p}(\Omega)} \leq b$ for every $n \in \mathbb{N}$.

For $\varepsilon > 0$, from Corollary 1.5.3, there exist $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset\subset \Omega$ with the cone property such that

$$||gu_n||_{L_s^p(\Omega)} \leq \frac{\varepsilon}{b}||u_n||_{W_s^{k,p}(\Omega)} + c(\varepsilon)||u_n||_{L^p(\Omega_\varepsilon)} \quad \forall n \in \mathbb{N}. \tag{1.33}$$

Since $W_s^{k,p}(\Omega) \subset W^{k,p}(\Omega_\varepsilon)$, we obtain the result from a well-known compact embedding theorem. □

Remark 1.5.5 : ***Comparing*** $\mathcal{G}(\Omega)$ ***and*** $\mathcal{C}^{\mathbf{k}}(\overline{\Omega})$

Difference: $\mathcal{C}^k(\overline{\Omega})$ *weights are more regular than* $\mathcal{G}(\Omega)$ *- functions, but these type of weights admit among their members a regularization function* $\sigma \in \mathcal{G}(\Omega) \cap C^\infty(\overline{\Omega})$ *of the same weight type but belonging to* $C^\infty(\overline{\Omega})$*, so more regular than a* $\mathcal{C}^k(\overline{\Omega})$ *function.*

Similarity: *Both admit a topological isomorphism, i.e. a map* $u \to \vartheta^s u$ *from* $W_s^{k,p}(\Omega)$ *to* $W^{k,p}(\Omega)$ *or from* $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ *to* $\overset{\circ}{W}{}^{k,p}(\Omega)$*, where* ϑ *is any weight function. For* $\mathcal{G}(\Omega)$ *class,* ϑ *is choosen as the regu-*

larization function σ, while for $\mathcal{C}^k(\overline{\Omega})$, it is just ρ, a $\mathcal{C}^k(\overline{\Omega})$ weight function.

Chapter 2

The Dirichlet problem in $\mathcal{G}(\Omega)$ - weighted Sobolev spaces on unbounded domains

In this chapter we prove an existence and uniqueness theorem for the following $\mathcal{G}(\Omega)$ - weighted problem

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega) \\ Lu = f, \ \ f \in L_s^p(\Omega) \end{cases} \tag{2.1}$$

where $s \in \mathbb{R}$, $p \in]1, +\infty[$, $W_s^{2,p}(\Omega)$, $\overset{\circ}{W}{}_s^{1,p}(\Omega)$ and $L_s^p(\Omega)$ are suitable $\mathcal{G}(\Omega)$ - weighted Sobolev spaces on an unbounded domain and L is the uniformly elliptic second order linear differential operator defined by

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a\,. \tag{2.2}$$

At this aim, using a general embedding result of section (1.5) about the multiplication operator

$$u \in W_s^{2,p}(\Omega) \to gu \in L_s^p(\Omega)$$

when g is a coefficient of L, we obtain some a priori estimates for the operator. Then, taking advantage of one of a priori bounds, an existence and uniqueness result in no - weighted spaces and the topological isomorphism (1.4.4), we are able to estabilish an existence and uniqueness theorem for weighted problem (2.1).

2.1 A priori estimates

Thanks to embedding results of section (1.5), we get two a priori estimates for the $\mathcal{G}(\Omega)$- Dirichlet problem. We recall that when Ω is bounded, several authors have been investigated the problem of determining a priori bounds under various hypotheses on the leading coefficients. It is worth to mention the results proved in [35], [19], [20], [55], [56], where the coefficients a_{ij} are required to be discontinuous. If the open set Ω is unbounded, a priori bounds are established in [51], [9] with analogous assumptions to those required in [35], while in [14], [10], [11], under similar hypotheses asked in [19], [20], the above estimates are obtained. Now, we extend some results of [19], [20] to a weighted case.

Assume that Ω is an unbounded open subset of $\mathbb{R}^n, n \geq 3$, with the uniform $C^{1,1}$-regularity property, $p \in]1, +\infty[$ and $s \in \mathbb{R}$.

Consider in Ω the differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \,, \tag{2.3}$$

with the following conditions on the coefficients:

$$(h_1) \qquad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{\text{loc}}(\bar{\Omega}) \,, \quad i, j = 1, \ldots, n \,, \\ \exists\, \nu > 0 \quad : \quad \displaystyle\sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \;\geq\; \nu |\xi|^2 \quad \text{a.e. in } \Omega \,, \; \forall\, \xi \in \mathbb{R}^n \,, \end{cases}$$

there exist functions e_{ij}, $i, j = 1, \ldots, n$, g and $\mu \in \mathbb{R}_+$ such that

$$(h_2) \qquad \begin{cases} e_{ij} = e_{ji} \in L^\infty(\Omega) \cap VMO(\Omega) \,, \quad i, j = 1, \ldots, n \,, \\ \displaystyle\sum_{i,j=1}^{n} e_{ij} \xi_i \xi_j \geq \mu |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall\, \xi \in \mathbb{R}^n \,, \\ g \in L^\infty(\Omega) \,, \quad \displaystyle\lim_{r \to +\infty} \sum_{i,j=1}^{n} ||e_{ij} - g\, a_{ij}||_{L^\infty(\Omega \setminus B_r)} = 0 \,, \end{cases}$$

$$(h_3) \qquad a_i \in \tilde{M}^{t_1}(\Omega), \; i = 1, \ldots, n \,, \quad a \in \tilde{M}^{t_2}(\Omega) \,,$$

where

$$t_1 \geq p, \quad t_1 \geq n, \quad t_1 > p \quad \text{if } \ p = n\,,$$

$$t_2 \geq p, \quad t_2 \geq n/2, \quad t_2 > p \quad \text{if } \ p = n/2\,.$$

Under assumptions (h_1) - (h_3), by Theorem 1.5.1, the operator $L : W^{2,p}_s(\Omega) \to L^p_s(\Omega)$ is bounded.

Let

$$L_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Theorem 2.1.1 *Suppose that assumptions $(h_1), (h_2)$ and (h_3) hold. Then there exist r_0, $c \in \mathbb{R}_+$ such that:*

$$||u||_{W^{2,p}_s(\Omega)} \leq c\big(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p_s(\Omega)}\big) \quad \forall\, u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega),$$

where c depends only on n, p, t_1, t_2, Ω, ν, μ, $||a_{ij}||_{L^\infty(\Omega)}, ||e_{ij}||_{L^\infty(\Omega)}, ||g||_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0} a_{ij}], \eta[e_{ij}], \widetilde{\sigma}[a_i], \widetilde{\sigma}[a], m, s,$ and r_0 depends only on n, p, Ω, μ, $||e_{ij}||_{L^\infty(\Omega)}$, $\eta[e_{ij}]$.

Proof $-$ Let $u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)$. By Lemma 1.4.4 we have that

$$\sigma^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega).$$

Then, by Theorem 3.1 of [10], there exist r_0 and $c_0 \in \mathbb{R}_+$ such that

$$||\sigma^s u||_{W^{2,p}(\Omega)} \leq c_0 \Big(||L_0(\sigma^s u)||_{L^p(\Omega)} + ||\sigma^s u||_{L^p(\Omega)} \Big), \tag{2.4}$$

where c_0 depends on $n, p, \Omega, \nu, \mu, ||a_{ij}||_{L^\infty(\Omega)}, ||e_{ij}||_{L^\infty(\Omega)}, ||g||_{L^\infty(\Omega)}, \eta[\zeta_{2r_0} a_{ij}]$, $\eta[e_{ij}]$, and r_0 depends on n, p, Ω, μ, $||e_{ij}||_{L^\infty(\Omega)}$, $\eta[e_{ij}]$. Since

$$\begin{aligned} L_0(\sigma^s u) &= \sigma^s Lu - s(s-1)\sigma^{s-2} \sum_{i,j=1}^n a_{ij}\sigma_{x_i}\sigma_{x_j} u - 2s\sigma^{s-1} \sum_{i,j=1}^n a_{ij}\sigma_{x_i} u_{x_j} + \\ &- s\sigma^{s-1} \sum_{i,j=1}^n a_{ij}\sigma_{x_i x_j} u - \sigma^s \sum_{i=1}^n a_i u_{x_i} - \sigma^s a u \,, \end{aligned} \tag{2.5}$$

from (2.4) and (2.5) we have

$$\begin{aligned} ||\sigma^s u||_{W^{2,p}(\Omega)} \leq c_1 \big(||\sigma^s Lu||_{L^p(\Omega)} + ||\sigma^s u||_{L^p(\Omega)} + \\ + \sum_{i,j=1}^n ||\sigma^{s-2}\sigma_{x_i}\sigma_{x_j} u||_{L^p(\Omega)} + \sum_{i,j=1}^n ||\sigma^{s-1}\sigma_{x_i} u_{x_j}||_{L^p(\Omega)} + \\ + \sum_{i,j=1}^n ||\sigma^{s-1}\sigma_{x_i x_j} u||_{L^p(\Omega)} + \sum_{i=1}^n ||\sigma^s a_i u_{x_i}||_{L^p(\Omega)} + ||\sigma^s a u||_{L^p(\Omega)} \big), \end{aligned} \tag{2.6}$$

where c_1 depends on the same parameters as c_0 and on s.

By Theorem 4.7 of [3], for all $i = 1, ..., n$ we have:

$$||u_{x_i}||_{L^p_s(\Omega)} \leq c_2 \Big(||u_{xx}||^{\frac{1}{2}}_{L^p_s(\Omega)} ||u||^{\frac{1}{2}}_{L^p_s(\Omega)} + ||u||_{L^p_s(\Omega)} \Big), \tag{2.7}$$

where c_2 depends on Ω, m, n, p.

Moreover, from Corollary 1.5.2, for any $\varepsilon \in \mathbb{R}_+$ and $i = 1, \dots, n$ there exist $c_1(\varepsilon)$, $c_2(\varepsilon) \in \mathbb{R}_+$ such that:

$$||a_i u_{x_i}||_{L^p_s(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_1(\varepsilon)||u_{x_i}||_{L^p_s(\Omega)}\,, \tag{2.8}$$

$$||au||_{L^p_s(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_2(\varepsilon)||u||_{L^p_s(\Omega)}\,, \tag{2.9}$$

where $c_1(\varepsilon)$ depends on $\varepsilon, \Omega, n, p, t_1, \widetilde{\sigma}\,[a_i]$ and $c_2(\varepsilon)$ depends on $\varepsilon, \Omega, n, p$, $t_2, \widetilde{\sigma}\,[a]$.

From (2.6)-(2.9), Lemma 1.4.2 and Lemma 1.4.4, it follows

$$||u||_{W^{2,p}_s(\Omega)} \leq c_3\big(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p_s(\Omega)} + \varepsilon ||u||_{W^{2,p}_s(\Omega)} + \tag{2.10}$$

$$+ c_3(\varepsilon)(||u_{xx}||^{\frac{1}{2}}_{L^p_s(\Omega)}||u||^{\frac{1}{2}}_{L^p_s(\Omega)} + ||u||_{L^p_s(\Omega)})\big),$$

where c_3 depends on the same parameters as c_0 and on s, m, and $c_3(\varepsilon)$ depends on $\varepsilon, \Omega, n, p, t_1, t_2, \widetilde{\sigma}\,[a_i], \widetilde{\sigma}\,[a]$.

For $\varepsilon = \frac{1}{2c_3}$, from (2.10) we have

$$||u||_{W^{2,p}_s(\Omega)} \leq c_4\big(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p_s(\Omega)} + ||u_{xx}||^{\frac{1}{2}}_{L^p_s(\Omega)}||u||^{\frac{1}{2}}_{L^p_s(\Omega)}\big), \tag{2.11}$$

where c_4 depends on the same parameters as c_3 and on $t_1, t_2, \widetilde{\sigma}\,[a_i], \widetilde{\sigma}\,[a]$.

Using Young's inequality and (2.11), we get the result. $\square$

Now we carry on displaying a priori bound in which there is a bounded open set. This estimate will be useful in the sequel to state the existence of the solution of the problem (2.1).

Add the following assumptions on the coefficients of L and on the weight function:

$$(h_4) \qquad \begin{cases} (e_{ij})_{x_h} \in \, M_\circ^{t,n-t}(\Omega)\,, \quad \text{with } t \in]2,n]\,, \quad i,j,h = 1,\ldots,n\,, \\ a_i \in M_\circ^{t_1}(\Omega)\,, \quad i = 1,\ldots,n\,, \\ a = a' + b,\ a' \in M_\circ^{t_2}(\Omega),\ b \in L^\infty(\Omega),\ b_0 = \operatorname*{ess\,inf}_{\Omega} b > 0, \\ g_0 = \operatorname*{ess\,inf}_{\Omega} g > 0\,, \\ \displaystyle\lim_{|x|\to+\infty} \frac{\sigma_x + \sigma_{xx}}{\sigma} = 0\,, \end{cases}$$

where t_1 and t_2 are defined as in (h_3).

Theorem 2.1.2 *Suppose that assumptions $(h_1), (h_2)$ and (h_4) hold. Then there are a real positive number c and a bounded open $\Omega_1 \subset\subset \Omega$ with the cone property such that:*

$$||u||_{W_s^{2,p}(\Omega)} \leq c\bigg(||Lu||_{L_s^p(\Omega)} + ||u||_{L^p(\Omega_1)}\bigg) \quad \forall\ u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega),$$

where c and Ω_1 are dependent only on n, p, Ω, ν, μ, g_0, b_0, t, t_1, t_2, m, s, $||a_{ij}||_{L^\infty(\Omega)}$, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}$, $||b||_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0} a_{ij}]$, $\sigma_0[(e_{ij})_x]$, $\sigma_0[a_i]$, $\sigma_0[a']$.

Proof $-$ Let $u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)$. By Lemma 1.4.4 we have that

$$\sigma^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega).$$

Applying Theorem 3.3 of [11] to the operator $L_0 + b$, we have that there exist a real number $c_0 \in \mathbb{R}_+$ and an open bounded subset $\Omega_0 \subset \Omega$ with the cone property such that

$$||\sigma^s u||_{W^{2,p}(\Omega)} \leq c_0 \Big(||(L_0 + b)(\sigma^s u)||_{L^p(\Omega)} + ||\sigma^s u||_{L^p(\Omega_0)} \Big),$$

where c_0 and Ω_0 are dependent on $n, p, \Omega, \nu, \mu, g_0, b_0$, t, $||a_{ij}||_{L^\infty(\Omega)}$, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}, ||b||_{L^\infty(\Omega)}, \eta[\zeta_{2r_0} a_{ij}], \sigma_0[(e_{ij})_x]$, and r_0 depends on $n, p, \Omega, \mu, g_0, b_0$,t, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}$, $||b||_{L^\infty(\Omega)}, \sigma_0[(e_{ij})_x]$.

Proceeding as in the proof of Theorem 2.1.1, we have

$$\begin{aligned} ||u||_{W^{2,p}_s(\Omega)} \quad &\leq \quad c_1 \big(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p(\Omega_0)} + \sum_{i,j=1}^{n} ||\sigma^{s-2} \sigma_{x_i} \sigma_{x_j} u||_{L^p(\Omega)} + \\ &+ \quad \sum_{i,j=1}^{n} ||\sigma^{s-1} \sigma_{x_i} u_{x_j}||_{L^p(\Omega)} + \sum_{i,j=1}^{n} ||\sigma^{s-1} \sigma_{x_i x_j} u||_{L^p(\Omega)} + \\ &+ \quad \sum_{i=1}^{n} ||a_i u_{x_i}||_{L^p_s(\Omega)} + ||a' u||_{L^p_s(\Omega)} \big), \end{aligned} \tag{2.12}$$

where c_1 depends on the same parameters as c_0 and on m, s.

From Corollary 1.5.3 and (1.6) of [50] it follows that for any $\varepsilon \in \mathbb{R}_+$ and $i, j = 1, \ldots, n$ there exist $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon) \in \mathbb{R}_+$ and some bounded open subsets $\Omega_1(\varepsilon) \subset\subset \Omega$, $\Omega_2(\varepsilon) \subset\subset \Omega$, $\Omega_3(\varepsilon) \subset\subset \Omega$ with the cone property such that

$$||\sigma^{s-2}\sigma_{x_i}\sigma_{x_j}u||_{L^p(\Omega)} \leq \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_1(\varepsilon)||u||_{L^p(\Omega_1(\varepsilon))}\,, \tag{2.13}$$

$$||\sigma^{s-1}\sigma_{x_i}u_{x_j}||_{L^p(\Omega)} \leq \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_2(\varepsilon)||u_{x_j}||_{L^p(\Omega_2(\varepsilon))}\,, \tag{2.14}$$

$$||\sigma^{s-1}\sigma_{x_ix_j}u||_{L^p(\Omega)} \leq \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_3(\varepsilon)||u||_{L^p(\Omega_3(\varepsilon))}\,, \tag{2.15}$$

where $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon), \Omega_1(\varepsilon), \Omega_2(\varepsilon), \Omega_3(\varepsilon)$ are dependent on $\varepsilon, \Omega, n, p, m, s$.

Using again Corollary 1.5.3 and Theorem 4.7 of [3] we have that there exist $c_4(\varepsilon), c_5(\varepsilon) \in \mathbb{R}_+$ and bounded open sets $\Omega_4(\varepsilon) \subset\subset \Omega$, $\Omega_5(\varepsilon) \subset\subset \Omega$ with the cone property such that:

$$||a_iu_{x_i}||_{L^p_s(\Omega)} \leq \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_4(\varepsilon)||u_{x_i}||_{L^p(\Omega_4(\varepsilon))} \leq \tag{2.16}$$

$$\leq \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_4(\varepsilon)\big(||u_{xx}||^{\frac{1}{2}}_{L^p(\Omega_4(\varepsilon))}||u||^{\frac{1}{2}}_{L^p(\Omega_4(\varepsilon))} + ||u||_{L^p(\Omega_4(\varepsilon))}\big)\,,$$

$$||a^{'}u||_{L^p_s(\Omega)} \leq \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_5(\varepsilon)||u||_{L^p(\Omega_5(\varepsilon))}\,, \tag{2.17}$$

where $c_4(\varepsilon)$ and $\Omega_4(\varepsilon)$ depend on $\varepsilon, \Omega, n, p, m, s, t_1, \sigma_0[a_i]$ and $c_5(\varepsilon)$, and $\Omega_5(\varepsilon)$ depend on $\varepsilon, \Omega, n, p, m, s, t_2, \sigma_0[a']$.

From (2.12)-(2.17) and Young's inequality we have the result. $\square$

From the latter result we obtain that $L : W^{2,p}_s(\Omega) \to L^p_s(\Omega)$ is a semi-Fredholm operator, i.e. the kernel is finite dimensional and the range is closed (see Theorem 5.2 of [44]).

Let us approach introducing necessary tools to obtain existence and uniqueness of the problem (3.1).

At first of all, from now on, we will focus our attention on weight functions m in $\mathcal{G}(\Omega)$ such that:

$$\lim_{|x|\to+\infty} m(x) = +\infty \tag{2.18}$$

or

$$\lim_{|x|\to+\infty} m(x) = 0. \tag{2.19}$$

Without loss of generality, we can assume that only (2.18) holds. In fact, if the assumption (2.18) doesn't hold and then (2.19) holds we could give again the same proofs choosing like σ the regularization function of the function $\frac{1}{m}$.

2.2 Tools

Let fix a cutoff function $f \in C_\circ^\infty(\overline{\mathbb{R}}_+)$ such that

$$0 \le f \le 1, \quad f(t) = 1 \text{ if } t \in [0,1], \quad f(t) = 0 \text{ if } t \in [2,+\infty[. \tag{2.20}$$

Then we can define a sequence of functions $(\zeta_k)_{k\in\mathbb{N}}$ by

$$\zeta_k : x \in \Omega \longrightarrow f\left(\frac{\sigma(x)}{k}\right) \qquad \forall k \in \mathbb{N}. \tag{2.21}$$

If $\Omega_k = \{x \in \Omega \ : \ \ \sigma(x) < k\ \}$, we easily have, for every $k \in \mathbb{N}$, that

$$0 \leq \zeta_k \leq 1, \quad \zeta_k = 1 \text{ on } \overline{\Omega}_k, \quad \zeta_k = 0 \text{ on } \Omega \setminus \Omega_{2k}, \quad \zeta_k \in C_\circ^\infty(\overline{\Omega}). \quad (2.22)$$

Now we can show that suitably combining the functions ζ_k and σ, we can determine a sequence of functions $(\eta_k)_{k\in\mathbb{N}}$, whose elements play a fundamental role in the sequel.

Let us define, for every $k \in \mathbb{N}$,

$$\eta_k(x) = 2k\,\zeta_k(x) + (1 - \zeta_k(x))\sigma(x), \qquad x \in \Omega. \qquad (2.23)$$

Simple calculations show that

$$\begin{aligned} \sigma(x) &\leq \eta_k(x), && \text{if } x \in \overline{\Omega}_{2k} && (2.24) \\ \eta_k(x) &\leq (1 + c_k)\sigma(x), && \text{if } x \in \overline{\Omega}_{2k} && (2.25) \\ \sigma(x) &= \eta_k(x), && \text{if } x \in \Omega \setminus \Omega_{2k}, && (2.26) \end{aligned}$$

where $c_k \in \mathbb{R}_+$ depends only on k. So for any $k \in \mathbb{N}$, it holds that

$$\sigma \sim \eta_k \qquad (2.27)$$

and

$$\sigma^s \sim \eta_k^s \qquad \forall s \in \mathbb{R}. \qquad (2.28)$$

Moreover, for every $k \in \mathbb{N}$ the following estimates about derivatives hold

$$\begin{aligned}
\left(\frac{(\eta_k)_x}{\eta_k}\right)(x) &= \left(\frac{(\eta_k)_{xx}}{\eta_k}\right)(x) = 0, && \text{if } x \in \Omega_k \\
\left(\frac{(\eta_k)_x}{\eta_k}\right)(x) &\leq c_1 \left(\frac{\sigma_x}{\sigma}\right)(x), && \text{if } x \in \Omega \setminus \Omega_k \\
\left(\frac{(\eta_k)_{xx}}{\eta_k}\right)(x) &\leq c_2 \left(\frac{\sigma_x^2 + \sigma\sigma_{xx}}{\sigma^2}\right)(x), && \text{if } x \in \Omega \setminus \Omega_k,
\end{aligned}$$

and, more generally,

$$\left(\frac{(\eta_k)_x}{\eta_k}\right)(x) \leq c_3 \sup_{x\in\Omega\setminus\Omega_k} \left(\frac{\sigma_x}{\sigma}\right)(x) \qquad \forall x \in \Omega \tag{2.29}$$

$$\left(\frac{(\eta_k)_{xx}}{\eta_k}\right)(x) \leq c_4 \sup_{x\in\Omega\setminus\Omega_k} \left(\frac{\sigma_x^2 + \sigma\,\sigma_{xx}}{\sigma^2}\right)(x) \qquad \forall x \in \Omega \tag{2.30}$$

$$\tag{2.31}$$

with c_1, c_2, c_3 and c_4 independent of k.

Now, we are in the position to prove the uniqueness and the existenxce of the solution of the problem (2.1). We remark that we obtain an existence and uniqueness theorem in according to this schem: we start stating

- the ***uniqueness*** of the solution of the $\mathcal{G}(\Omega)$- Dirichlet problem deducing it from existence and uniqueness for the same but no-weighted problem

we carry on proving

- the ***existence*** of the solution applying the method of continuity along a parameter by means some tools as a weighted a priori bound, the topological isomorphism, some properties of regularization function.

2.3 A uniqueness result

Let assume that Ω is an unbounded open subset of $\mathbb{R}^n, n \geq 3$, with the uniform $C^{1,1}$-regularity property. Moreover, let $p \in]1, +\infty[$ and $s \in \mathbb{R}$. Consider in Ω the differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a \tag{2.32}$$

with the following conditions on the coefficients:

$$(h_1) \qquad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{\mathrm{loc}}(\bar{\Omega}) \,, \quad i,j = 1, \ldots, n \,, \\ \exists \, \nu > 0 \quad : \quad \displaystyle\sum_{i,j=1}^{n} a_{ij} \, \xi_i \, \xi_j \, \geq \, \nu |\xi|^2 \quad \text{a.e. in } \Omega \,, \; \forall \, \xi \in \mathbb{R}^n \,, \end{cases}$$

there exist functions e_{ij}, $i,j=1,\dots,n$, g and $\mu \in \mathbb{R}_+$ such that

$$(h_2')\qquad \begin{cases} e_{ij} = e_{ji} \in L^\infty(\Omega)\,, \quad i,j=1,\dots,n\,, \\ (e_{ij})_{x_h} \in M_\circ^{t,n-t}(\Omega)\,, \quad \text{with } t \in]2,n]\,, \quad i,j,h=1,\dots,n\,, \\ \displaystyle\sum_{i,j=1}^n e_{ij}\xi_i\xi_j \geq \mu|\xi|^2 \ \text{ a.e. in } \Omega\,, \quad \forall\, \xi \in \mathbb{R}^n\,, \\ g \in L^\infty(\Omega)\,, \quad g_0 = \operatorname*{ess\,inf}_\Omega g > 0\,, \quad g \in \mathrm{Lip}(\overline{\Omega})\,, \\ \displaystyle\lim_{r\to+\infty} \sum_{i,j=1}^n ||e_{ij} - g\, a_{ij}||_{L^\infty(\Omega\setminus B_r)} = 0\,, \end{cases}$$

$$(h_3')\qquad \begin{cases} a_i \in M_\circ^{t_1}(\Omega)\,,\ i=1,\dots,n\,, \\ a = a' + b\,,\ a' \in M_\circ^{t_2}(\Omega)\,,\ b \in L^\infty(\Omega)\,,\ b_0 = \operatorname*{ess\,inf}_\Omega b > 0\,, \\ a_0 = \operatorname*{ess\,inf}_\Omega a > 0\,, \end{cases}$$

where

$$t_1 > n \quad \text{if } p \leq n\,, \qquad t_1 = p \quad \text{if } p > n\,,$$

$$t_2 > n/2 \quad \text{if } p \leq n/2\,, \qquad t_2 = p \quad \text{if } p > n/2\,.$$

Adding the following assumption on the weight function

$$(h_4')\qquad \lim_{k\to+\infty} \sup_{\Omega\setminus\Omega_k} \frac{\sigma_x + \sigma_{xx}}{\sigma} = 0\,,$$

we can prove our uniqueness theorem.

Theorem 2.3.1 *Assume* (h_1) – $(h_4^{'})$ *true. Then the problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega) \\ Lu = 0 \end{cases} \tag{2.33}$$

has only the zero solution.

Proof – From Theorem 4.3 of [11] and from the bounded inverse theorem (see Theorem 3.8 of [44]), there exists $c_1 \in \mathbb{R}_+$ such that

$$||u||_{W^{2,p}(\Omega)} \leq c_1 ||Lu||_{L^p(\Omega)} \qquad \forall u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega) \,. \tag{2.34}$$

Fix $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)$. Since $\eta_k^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$ $\forall k \in \mathbb{N}$ (see Lemma 3.4 of [4]), from (2.34) then there exists $c_2 \in \mathbb{R}_+$, independent of u and k, such that

$$||\eta_k^s u||_{W^{2,p}(\Omega)} \leq c_2 ||L(\eta_k^s u)||_{L^p(\Omega)}. \tag{2.35}$$

For simplicity, in the sequel, we will write $\eta_k = \eta$. Since

$$\begin{aligned} L(\eta^s u) &= \eta^s Lu - s \sum_{i,j=1}^{n} a_{ij} \Big((s-1)\eta^{s-2}\eta_{x_i}\eta_{x_j} u + \eta^{s-1}\eta_{x_i x_j} u + \\ &+ 2\eta^{s-1}\eta_{x_i} u_{x_j} \Big) + s \sum_{i=1}^{n} a_i \eta^{s-1}\eta_{x_i} u \,, \end{aligned} \tag{2.36}$$

from (2.35) and (2.36) we have:

$$\begin{aligned} ||\eta^s\, u||_{W^{2,p}(\Omega)} \;&\leq\; c_3\Bigg(||\eta^s\, Lu||_{L^p(\Omega)} + \sum_{i,j=1}^{n}\big(||\eta^{s-2}\eta_{x_i}\eta_{x_j}u||_{L^p(\Omega)} + \\ &+\; ||\eta^{s-1}\eta_{x_ix_j}u||_{L^p(\Omega)} + ||\eta^{s-1}\eta_{x_i}u_{x_j}||_{L^p(\Omega)}\big) + \\ &+\; \sum_{i=1}^{n}||a_i\eta^{s-1}\eta_{x_i}u||_{L^p(\Omega)}\Bigg)\,, \end{aligned} \tag{2.37}$$

where $c_3 \in \mathbb{R}_+$ is independent of u and k. From Theorem 1.5.1 with $s = 0$ and from (2.29) we get:

$$||a_i\eta^{s-1}\eta_{x_i}u||_{L^p(\Omega)} \leq c_4 \sup_{\Omega\setminus\Omega_k}\frac{\sigma_x}{\sigma}\; ||a_i||_{M^{t_1}(\Omega)}||\eta^s u||_{W^{1,p}(\Omega)}\,, \tag{2.38}$$

where c_4 is independent of u and k.

Thus, by (2.29), (2.30), (2.37) and (2.38), with easy computations, we obtain the bound:

$$\begin{aligned} ||\eta^s\, u||_{W^{2,p}(\Omega)} \;&\leq\; c_5\Bigg[||\eta^s\, Lu||_{L^p(\Omega)} + \Bigg(\sup_{\Omega\setminus\Omega_k}\frac{\sigma_x^2+\sigma\,\sigma_{xx}}{\sigma^2} + \\ &+\; \sup_{\Omega\setminus\Omega_k}\frac{\sigma_x}{\sigma}\Bigg)\; ||\eta^s\, u||_{W^{2,p}(\Omega)}\Bigg], \end{aligned} \tag{2.39}$$

where c_5 is independent of u and k.

By hypothesis (h_4'), there exists $k_0 \in \mathbb{N}$ such that:

$$\Bigg(\sup_{\Omega\setminus\Omega_{k_0}}\frac{\sigma_x^2+\sigma\,\sigma_{xx}}{\sigma^2} + \sup_{\Omega\setminus\Omega_{k_0}}\frac{\sigma_x}{\sigma}\Bigg) \leq \frac{1}{2\,c_5}\,. \tag{2.40}$$

Now, if we denote with η the function η_{k_0}, from (2.39) and (2.40) we can deduce that:

$$||\eta^s\, u||_{W^{2,p}(\Omega)} \leq c_6 ||\eta^s\, Lu||_{L^p(\Omega)}\,, \tag{2.41}$$

and then, using (2.28), from (2.41) we obtain that:

$$||u||_{W^{2,p}_s(\Omega)} \leq c_7 ||Lu||_{L^p_s(\Omega)}\,, \tag{2.42}$$

with c_6, c_7 independent of u, and then the claimed result. $\square$

2.4 Existence results

The aim of this section is to establish some existence results concerning the problem

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega) \\ Lu = f,\ \ f \in L^p_s(\Omega). \end{cases} \tag{2.43}$$

We start with a lemma which we will need in the proof of our main existence result.

Lemma 2.4.1 *Let*

$$L_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

and assume that (h_1),$(h_2^{'})$,$(h_4^{'})$ *hold. Then the Dirichlet problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega) \\ L_0 u + cu = f\,, \quad f \in L_s^p(\Omega) \end{cases} \tag{2.44}$$

where

$$c = 1 + \Big| - s(s+1) \sum_{i,j=1}^{n} a_{ij} \frac{\sigma_{x_i}}{\sigma} \frac{\sigma_{x_j}}{\sigma} + s \sum_{i,j=1}^{n} a_{ij} \frac{\sigma_{x_i x_j}}{\sigma} \Big|\,, \tag{2.45}$$

is uniquely solvable.

PROOF − Note that u is a solution of the problem (2.44) if and only if $w = \sigma^s u$ is a solution of the problem

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega) \\ -\displaystyle\sum_{i,j=1}^{n} a_{ij} (\sigma^{-s} w)_{x_i x_j} + c\sigma^{-s} w = f\,, \quad f \in L_s^p(\Omega)\,. \end{cases} \tag{2.46}$$

Since, for any $i, j \in \{1, \dots, n\}$

$$\begin{aligned} \frac{\partial^2}{\partial x_i \partial x_j} (\sigma^{-s} w) &= \sigma^{-s} w_{x_i x_j} - 2s\sigma^{-s-1} \sigma_{x_i} w_{x_j} + s(s+1)\sigma^{-s-2} \sigma_{x_i} \sigma_{x_j} w + \\ &- s\sigma^{-s-1} \sigma_{x_i x_j} w, \end{aligned}$$

then (2.46) is equivalent to the problem

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega) \\ L_0 w + \sum_{i=1}^{n} \alpha_i w_{x_i} + \alpha w = \sigma^s f \end{cases} \tag{2.47}$$

where:

$$\alpha_i = 2s \sum_{j=1}^{n} a_{ij} \frac{\sigma_{x_j}}{\sigma}\,, \qquad i = 1, \ldots, n,$$

$$\alpha = c - s(s+1) \sum_{i,j=1}^{n} a_{ij} \frac{\sigma_{x_i}}{\sigma} \frac{\sigma_{x_j}}{\sigma} + s \sum_{i,j=1}^{n} a_{ij} \frac{\sigma_{x_i x_j}}{\sigma} \quad .$$

By Theorem 4.3 of [11], (1.6) of [50] and (1.24), we obtain that (2.47) is uniquely solvable and then the problem (2.44) is uniquely solvable too.

□

Theorem 2.4.2 *Suppose that conditions* (h_1) – $(h_4^{'})$ *hold. Then the problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega) \\ Lu = f\,, \quad f \in L_s^p(\Omega) \end{cases} \tag{2.48}$$

is uniquely solvable.

Proof − For each $\tau \in [0,1]$ put

$$L_\tau = \tau L + (1-\tau)(L_0 + c)\,,$$

where c is the function defined by (2.45). The operator

$$\tau \in [0,1] \longmapsto L_\tau \in B(W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega), L_s^p(\Omega))$$

is clearly continuous. By Theorem 5.2 of [4] and Theorem 2.3.1 we can say that the operator L_τ has closed range and null kernel. Now, by Lemma 4.1 of [11], there exists a positive real number c_0 such that

$$\begin{gathered} ||u||_{W_s^{2,p}(\Omega)} \leq c_0 ||L_\tau u||_{L_s^p(\Omega)}\,, \\ \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)\,, \quad \forall \tau \in [0,1]. \end{gathered} \tag{2.49}$$

Using the Lemma 2.4.1, the problem

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega) \\ L_0 u + cu = f\,, \quad f \in L_s^p(\Omega) \end{cases} \tag{2.50}$$

is uniquely solvable.

Therefore, this latter result and the estimate (2.49) allow to use the method of continuity along a parameter (see, e.g., Theorem 5.2 of [23]) in order to prove that the problem

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega) \\ Lu = f\,, \quad f \in L_s^p(\Omega) \end{cases} \tag{2.51}$$

is likewise uniquely solvable. The proof is now complete. □

Chapter 3

The Dirichlet problem in $\mathcal{G}(\Omega)$ - weighted Sobolev spaces on unbounded domains of the plane

Here, we deal with existence and uniqueness results for solution of the Dirichlet problem weighted with $\mathcal{G}(\Omega)$ - functions in unbounded domains of the plane. Actually, we consider the following Dirichlet problem:

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)\,, \\ Lu = f\,, \quad f \in L^p_s(\Omega)\,, \end{cases} \tag{3.1}$$

where $s \in \mathbb{R}$, $p \in]1, +\infty[$, $W^{2,p}_s(\Omega)$, $\overset{\circ}{W}{}^{1,p}_s(\Omega)$ and $L^p_s(\Omega)$ are suitable weighted Sobolev spaces on an unbounded domains in $\mathbb{R}^2$. Our first purpose is to collect the recent contributions to the $W^{2,p}-$ solvability in domains in $\mathbb{R}^2$, bounded as well unbounded, for any value of p in the range $]1,+\infty[$, of the no weighted problem:

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \\ Lu = f\,, \quad f \in L^p(\Omega)\,, \end{cases} \tag{3.2}$$

(see [15, 16, 17]).

3.1 $W^{2,p}$ - solvability in bounded planar domains

Let Ω be a bounded $C^{1,1}$ - open subset of $\mathbb{R}^2$ and let $p \in]1,+\infty[$. Consider in Ω the uniformly elliptic second order linear differential operator

$$L = -\sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} + a\,, \tag{3.3}$$

and the following hypotheses on its coefficients:

$$(h_1) \quad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO(\Omega), \quad i,j = 1,2\,, \\ \exists\, \nu \in \mathbb{R}_+ : \displaystyle\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \qquad \text{a.e. in } \Omega, \quad \forall\, \xi \in \mathbb{R}^2\,; \end{cases}$$

$$(H_2) \qquad \begin{cases} a_i \in L^r(\Omega), \ i = 1, 2, \\ \text{where} \quad r > 2 \quad \text{if} \ p \leq 2\,, \quad r = p \ \text{if} \ p > 2\,, \\ a \in L^p(\Omega)\,. \end{cases}$$

Then, by Sobolev embedding theorem, the linear operator L defined in $W^{2,p}(\Omega)$ attains its values into $L^p(\Omega)$ and it is bounded. Moreover, as proved in [15], one also infers an a priori estimate, some regularity properties and the solvability result. We just list them without proofs.

Lemma 3.1.1 *Under (h_1) and (H_2), then a positive constant c exists such that*

$$||u||_{W^{2,p}(\Omega)} \leq c\left(||Lu||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}\right) \qquad \forall u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega),$$

c depends on Ω, p, ν, $||a_{ij}||_{L^\infty(\Omega)}$, $\eta[p(a_{ij})]$, $||a_i||_{L^r(\Omega)}$, $||a||_{L^p(\Omega)}$, $\omega_r[a_i]$, $\omega_p[a]$, where $p(a_{ij})$ is an extension of a_{ij} to $\mathbb{R}^2$ of class $L^\infty(\mathbb{R}^2) \cap VMO(\mathbb{R}^2)$.

Lemma 3.1.2 *Under (h_1) and (H_2), then any solution u of the problem*

$$\begin{cases} u \in W^{2,q}(\Omega) \cap \overset{\circ}{W}{}^{1,q}(\Omega)\,, \ \textit{with} \ \ q \leq p, \\ Lu \in L^p(\Omega)\,, \end{cases}$$

belongs to $W^{2,p}(\Omega)$.

Theorem 3.1.3 *Under (h_1) and (H_2), if $essinf_\Omega a \geq 0$, then problem (3.2) is uniquely solvable in $W^{2,p}(\Omega)$ and the solution u satisfies the a* *priori bound*

$$||u||_{W^{2,p}(\Omega)} \leq c||f||_{L^p(\Omega)},$$

with $c \in \mathbb{R}_+$ *depending on* Ω, p, ν, $||a_{ij}||_{L^\infty(\Omega)}$, $\eta[p(a_{ij})]$, $||a_i||_{L^r(\Omega)}$, $||a||_{L^p(\Omega)}$, $\omega_r[a_i]$, $\omega_p[a]$ *and where* $p(a_{ij})$ *is the extension of* a_{ij} *to* $\mathbb{R}^2$ *considered in Lemma 3.1.2.*

3.2 $W^{2,p}$- solvability in unbounded planar domains

Now let Ω be an unbounded uniformly-$C^{1,1}$ open set in $\mathbb{R}^2$ and, as above, let $p \in]1, +\infty[$. Consider the differential operator L defined in (3.3) and the following hypotheses on its coefficients:

$$(h_1') \quad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{\text{loc}}(\bar{\Omega}), \quad i, j = 1, 2\,, \\ \exists\, \nu \in \mathbb{R}_+ : \displaystyle\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \qquad \text{a.e. in } \Omega, \quad \forall\, \xi \in \mathbb{R}^2\,; \end{cases}$$

there exist functions e_{ij} and g and a constant $\mu \in \mathbb{R}_+$ s. t.

$$(h_1'') \qquad \begin{cases} e_{ij} = e_{ji} \in L^\infty(\Omega) \cap VMO(\Omega)\,, \quad i, j = 1, 2\,, \\ \displaystyle\sum_{i,j=1}^{2} e_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2 \ \text{ a.e. in } \Omega, \ \ \forall \xi \in \mathbb{R}^2\,, \\ g \in L^\infty(\Omega), \\ \displaystyle\lim_{\rho \to +\infty} \sum_{i,j=1}^{2} ||e_{ij} - g\, a_{ij}||_{L^\infty(\Omega \setminus B_\rho)} = 0; \end{cases}$$

$$
(H_2') \qquad \begin{cases} a_i \in \tilde{M}^r(\Omega), \; i = 1, 2, \\ \text{where } \; r > 2 \quad \text{if } \; p \le 2\,, \quad r = p \; \text{ if } \; p > 2\,, \\ a \in \tilde{M}^p(\Omega)\,. \end{cases}
$$

We like to stress that assumptions (h_1')-(h_1'') are weaker than the one express by (h_1) above when the underlying domain Ω is unbounded, as exhibited in Section 6 of [10].

First we report an a priori estimate for solutions to (3.2) (see [17], Theorem 3.2), by determining suitable localizations of the stated problem in order to apply Lemma (3.1.2).

Lemma 3.2.1 *Under (h_1')-(h_1'') and (H_2'), then there exist positive real numbers ρ_0, c such that*

$$
||u||_{W^{2,p}(\Omega)} \le c \left(||Lu||_{L^p(\Omega)} + ||u||_{L^p(\Omega)} \right) \qquad \forall u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega),
$$

with c depending only on Ω, p, r, ν, μ, $||a_{ij}||_{L^\infty(\Omega)}$, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}$, $\eta[p(\zeta_{2\rho_0} a_{ij})]$, $\eta[p(e_{ij})]$, $\tilde{\sigma}_r[a_i]$, $\tilde{\sigma}_p[a]$.

Moreover, the following global regularity result holds

Lemma 3.2.2 *Under (h_1')-(h_1'') and (H_2'), if u is a solution of the problem*

$$
\begin{cases} u \in W^{2,q}_{loc}(\bar{\Omega}) \cap \overset{o}{W}{}^{1,q}_{loc}(\bar{\Omega}) \cap L^{q_o}(\Omega), \;\; \textit{with } q \in \,]1, p], \; q_o \in [1, p], \\ Lu \, \in \, L^p(\Omega), \end{cases}
$$

then u belongs to $W^{2,p}(\Omega)$.

It is now possible to give answer to the strong solvability of (3.2). In order to prove the uniqueness result is however necessary to handle with a suitable maximum principle, established in [16] for arbitrary domains of $\mathbb{R}^n$, $n \geq 2$. It is well known, in fact, that the classical Aleksandrov-Bakel'man-Pucci principle requires the solution to belong to $W^{2,n}_{\mathrm{loc}}(\Omega) \cap C^o(\overline{\Omega})$.

Since in this case the assumptions on the coefficients are much weaker, we prefer to write them down as

$$(h_M) \quad \begin{cases} a_{ij} = a_{ji} \in L^{\infty}_{\mathrm{loc}}(\Omega) \cap VMO_{\mathrm{loc}}(\Omega)\,, \quad i,j = 1,\dots,n\,, \\ a_i \in L^{r}_{\mathrm{loc}}(\Omega), i = 1,\dots,n, \\ \text{where } r > n \;\text{ if } p \leq n, r = p \text{ if } \; p > n, \\ a \in L^{p}_{\mathrm{loc}}(\Omega)\,, \\ \exists \nu \in L^{\infty}_{\mathrm{loc}}(\Omega) : \nu(x) > 0 \quad \text{a.e. in } \; \Omega\,, \\ \displaystyle\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \; \nu(x)|\xi|^2 \;\text{ a.e. in } \; \Omega\,, \;\; \forall \xi \in \mathbb{R}^n\,, \\ \text{for any open subset } E \subset\subset \Omega\,, \;\; \mathrm{essinf}_E \nu > 0\,, \;\; \mathrm{essinf}_E a > 0. \end{cases}$$

Then we mention the following result (see [16], Theorem 4.1)

Theorem 3.2.3 *Let Ω be an arbitrary open set in $\mathbb{R}^n$, $n \geq 2$. Suppose that $p > \frac{n}{2}$ and (h_M) holds. If u is a solution of the problem*

$$u \in W^{2,p}_{loc}(\Omega)\,, \quad Lu \geq \; 0\,,$$

then u does not have any positive relative maximum in Ω.

As consequence it has been deduced

Corollary 3.2.4 *Under the assumption of Theorem 3.2.3, the problem*

$$\begin{cases} u \in W^{2,p}_{loc}(\Omega)\,, \quad Lu = 0\,, \\ \lim\limits_{x \to x_o} u(x) = 0 \qquad \forall x_o \in \partial\Omega\,, \\ \lim\limits_{|x| \to +\infty} u(x) = 0 \qquad \textit{if } \Omega \textit{ is unbounded,} \end{cases}$$

has only the zero solution.

Hence we are now in position to show contributions to the study of strong solvability of (3.2) in unbounded planar domains in the following two theorems contained in [17]. We begin with the uniqueness result, which turns out combining the regularity property of the differential operator L proved in Lemma (3.2.2) with the previous Corollary (3.2.4).

Theorem 3.2.5 *Assume $(h_1'), (H_2')$ and $a \geq a_0$ a.e. in Ω for some $a_0 \in \mathbb{R}_+$; if $p \leq 2$, suppose also (h_1''). Then the problem*

$$(\mathcal{D}) \qquad \begin{cases} u \in W^{2,p}_{loc}(\bar{\Omega}) \cap W^{1,p}_o(\Omega)\,, \\ Lu = 0, \end{cases}$$

admits only the zero solution in Ω.

Assuming

$$(h'_E)\quad \begin{cases} (e_{ij})_{x_h}\,,\ a_i \in M^r_o(\Omega)\,,\quad i,j,h=1,2\,,\\[1em] \text{where } r>2 \ \text{ if } \ p\le 2\,,\quad r=p \ \text{ if } \ p>2\,,\\[1em] a=a'+b, \text{ where } a'\in M^p_o(\Omega),\ b\in L^\infty(\Omega),\ b_o=\mathrm{essinf}_\Omega b>0,\\[1em] g\in \mathrm{Lip}(\bar\Omega)\,,\quad g_o=\mathrm{essinf}_\Omega g>0\,, \end{cases}$$

we conclude establishing

Theorem 3.2.6 *If (h'_1), (h'_E) hold and $a \geq a_0$ a.e. in Ω for some $a_0 \in \mathbb{R}_+$, then the Dirichlet problem*

$$(\mathcal{D}_p)\qquad \begin{cases} u\in W^{2,p}(\Omega)\cap \overset{\circ}{W}{}^{1,p}(\Omega)\,,\\[1em] Lu=f\,,\qquad f\in L^p(\Omega)\,, \end{cases}$$

is uniquely solvable.

Remark 3.2.7 *In order to illustrate that assumptions of Theorem 3.2.6 does not imply $(a_{ij})_{x_h}$ to be into $M^p_o(\Omega)$, we sketch the following example.*

Let $\Omega :=]-\infty,\infty[\times]-1,1[\subset \mathbb{R}^2$. Define $\alpha_{ij} := 2\delta_{ij}$ and

$$a_{ij} := \alpha_{ij} + \frac{\sin\left(1+e^{|x|^2}\right)}{1+|x|}\,\delta_{ij}\,,\quad i,j=1,2.$$

Then the functions a_{ij} verify the assumptions (h'_1), (h'_E), whereas $(a_{ii})_{x_h}$ do not belong to $M^p_o(\Omega)$ for any $p\in[1,+\infty[$ and $i,h=1,2$.

Now, we are ready to introduce our results about $W^{2,p}_s$ - solvability on unbounded domains of the plane. At this aim, we start stating

3.3 A $\mathcal{G}(\Omega)$ - weigthed a priori estimate

Let Ω be an unbounded open subset of $\mathbb{R}^2$, with the uniform $C^{1,1}$-regularity property, and let $p \in]1, +\infty[$, $s \in \mathbb{R}$. Consider in Ω the differential operator L (3.3) with the following conditions on the coefficients:

$$(h_1') \quad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{loc}(\bar{\Omega}) \,, \quad i,j = 1,2 \,, \\ \exists\, \nu > 0 \quad : \quad \displaystyle\sum_{i,j=1}^{2} a_{ij}\, \xi_i\, \xi_j \,\geq\, \nu |\xi|^2 \qquad \text{a.e. in } \Omega \,,\ \forall \xi \in \mathbb{R}^2 \,, \end{cases}$$

there exist functions e_{ij}, $i,j = 1,2$, g and $\mu \in \mathbb{R}_+$ such that

$$(h_2) \quad \begin{cases} e_{ij} = e_{ji} \in L^\infty(\Omega) \,,\ (e_{ij})_{x_h} \in M^t_\circ(\Omega) \,, \quad i,j,h = 1,2 \,, \\ \displaystyle\sum_{i,j=1}^{2} e_{ij} \xi_i \xi_j \,\geq\, \mu |\xi|^2 \qquad \text{a.e. in } \Omega \,, \quad \forall\, \xi \in \mathbb{R}^2 \,, \\ g \in L^\infty(\Omega) \,, \qquad \displaystyle\lim_{r \to +\infty} \sum_{i,j=1}^{2} ||e_{ij} - g\, a_{ij}||_{L^\infty(\Omega \setminus B_r)} = 0 \,, \\ g \in Lip(\bar{\Omega}) \,,\ \ g_0 = \operatorname*{ess\,inf}_{\Omega} g > 0 \,, \end{cases}$$

$$(h_3) \quad \begin{cases} a_i \in M^t_\circ(\Omega) \,,\ i = 1,2 \,, \\ a = a' + b \,,\ a' \in M^p_\circ(\Omega) \,,\ b \in L^\infty(\Omega) \,,\ b_0 = \operatorname*{ess\,inf}_{\Omega} b > 0 \,, \end{cases}$$

where

$$t > 2 \quad \text{if} \quad p \leq 2\,, \quad t = p \quad \text{if} \quad p > 2\,.$$

Let us fix $m \in \mathcal{G}(\Omega)$ such that (2.18) and

$$(h_4) \qquad \lim_{|x| \to +\infty} \frac{\sigma_x + \sigma_{xx}}{\sigma} = 0$$

hold.

We are able to prove the following a priori estimate.

Theorem 3.3.1 *Suppose that the hypotheses (h_1') - (h_4) hold. Then there are a positive constant c_0 and a bounded open subset $\Omega_0 \subset\subset \Omega$ with the cone property such that:*

$$||u||_{W^{2,p}_s(\Omega)} \leq c_0 \left(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p(\Omega_0)}\right) , \quad \forall\, u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)\,. \tag{3.4}$$

Proof — Notice that the boundedness of the operator $L : W^{2,p}_s(\Omega) \to L^p_s(\Omega)$ follows from Theorem 1.5.1.

Denote by L_0 the principal part of the operator, that is

$$L_0 = -\sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}\,.$$

Let us fix $u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)$. By means of the topological isomorphism (1.4.4) we have that

$$\sigma^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,.$$

Applying Theorem 5.2 of [17] and the bounded inverse theorem (see Theorem 3.8 of [44]) to the operator $L_0 + b$, we get

$$||\sigma^s u||_{W^{2,p}(\Omega)} \leq c_1 \left(||(L_0 + b)(\sigma^s u)||_{L^p(\Omega)}\right)\,,$$

where c_1 is a constant independent of u. Using again the topological isomorphism (1.4.4), with simple calculations, we have:

$$||u||_{W^{2,p}_s(\Omega)} \leq c_2 \Big(||Lu||_{L^p_s(\Omega)} + \sum_{i,j=1}^{2} \left(||\sigma_{x_i}\sigma_{x_j}\sigma^{-2}u||_{L^p_s(\Omega)} + ||\sigma_{x_i}\sigma^{-1}u_{x_j}||_{L^p_s(\Omega)} + \right.$$

$$\left. + ||\sigma_{x_i x_j}\sigma^{-1}u||_{L^p_s(\Omega)}\right) + \sum_{i=1}^{2} ||a_i u_{x_i}||_{L^p_s(\Omega)} + ||a' u||_{L^p_s(\Omega)}\Big), \quad (3.5)$$

where c_2 is independent of u. From Corollary 1.5.3 and (1.6) in [50] we deduce that for any $\varepsilon \in \mathbb{R}_+$ and $i, j = 1, 2$ there exist $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon) \in \mathbb{R}_+$ and some bounded open subsets $\Omega_1(\varepsilon)$, $\Omega_2(\varepsilon)$, $\Omega_3(\varepsilon) \subset\subset \Omega$ with the cone property such that

$$||\sigma_{x_i}\sigma_{x_j}\sigma^{-2}u||_{L^p_s(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_1(\varepsilon)||u||_{L^p(\Omega_1(\varepsilon))}\,, \quad (3.6)$$

$$||\sigma_{x_i}\sigma^{-1}u_{x_j}||_{L^p_s(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_2(\varepsilon)||u_{x_j}||_{L^p(\Omega_2(\varepsilon))}\,, \quad (3.7)$$

$$||\sigma_{x_i x_j}\sigma^{-1}u||_{L^p_s(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_3(\varepsilon)||u||_{L^p(\Omega_3(\varepsilon))}\,, \quad (3.8)$$

where $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon), \Omega_1(\varepsilon), \Omega_2(\varepsilon), \Omega_3(\varepsilon)$ are dependent only on ε, Ω, p, m, s.

Applying again Corollary 1.5.3 we have that there exist $c_4(\varepsilon), c_5(\varepsilon) \in \mathbb{R}_+$ and some bounded open subsets $\Omega_4(\varepsilon)$, $\Omega_5(\varepsilon) \subset\subset \Omega$ with the cone property such that:

$$||a_i u_{x_i}||_{L^p_s(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_4(\varepsilon)||u_{x_i}||_{L^p(\Omega_4(\varepsilon))}\,, \tag{3.9}$$

$$||a' u||_{L^p_s(\Omega)} \leq \varepsilon ||u||_{W^{2,p}_s(\Omega)} + c_5(\varepsilon)||u||_{L^p(\Omega_5(\varepsilon))}\,, \tag{3.10}$$

where $c_4(\varepsilon)$ and $\Omega_4(\varepsilon)$ depend on $\varepsilon, \Omega, p, m, s, t, \sigma_0[a_i]$, and $c_5(\varepsilon)$ and $\Omega_5(\varepsilon)$ depend on $\varepsilon, \Omega, p, m, s, t, \sigma_0[a']$.

Combining the above estimates (3.5) - (3.10), we obtain

$$\begin{aligned} ||u||_{W^{2,p}_s(\Omega)} \leq c_3\Big(& ||Lu||_{L^p_s(\Omega)} + \varepsilon ||u||_{W^{2,p}_s(\Omega)} + \\ & + c_6(\varepsilon)\big(||u||_{L^p(\Omega_6(\varepsilon))} + ||u_x||_{L^p(\Omega_6(\varepsilon))}\big)\Big)\,, \end{aligned} \tag{3.11}$$

where c_3 is independent of u, $c_6(\varepsilon)$ and $\Omega_6(\varepsilon)$ depend only on $\varepsilon, \Omega, p, m, s$, $t, \sigma_0[a_i], \sigma_0[a']$.

On the other hand, by the Gagliardo - Nirenberg inequality

$$||u_x||_{L^p(\Omega_6(\varepsilon))} \leq c_7(\varepsilon)\Big(||u_{xx}||^{\frac{1}{2}}_{L^p(\Omega_6(\varepsilon))}||u||^{\frac{1}{2}}_{L^p(\Omega_6(\varepsilon))} + ||u||_{L^p(\Omega_6(\varepsilon))}\Big)\,, \tag{3.12}$$

with $c_7(\varepsilon) \in \mathbb{R}_+$ dependent on ε, Ω and p. So (3.11), (3.12) and (1.20) lead to:

$$||u||_{W^{2,p}_s(\Omega)} \leq c_3\Big(||Lu||_{L^p_s(\Omega)} + \varepsilon||u||_{W^{2,p}_s(\Omega)} +$$
$$+ c_8(\varepsilon)(||u_{xx}||^{\frac{1}{2}}_{L^p_s(\Omega_6(\varepsilon))}||u||^{\frac{1}{2}}_{L^p_s(\Omega_6(\varepsilon))} + ||u||_{L^p(\Omega_6(\varepsilon))})\Big)\,, \quad (3.13)$$

with $c_8(\varepsilon) \in \mathbb{R}_+$ dependent on $\varepsilon, \Omega, p, m, s, t, \sigma_0[a_i], \sigma_0[a']$.

Now, if we choose $\varepsilon = \frac{1}{2c_3}$ and use the Young's inequality, from (3.13) we get the result. □

Now, we can display

3.4 $W^{2,p}_s$-solvability on unbounded domains of the plane

We begin this section with the uniqueness theorem for the homogeneous Dirichlet problem in the plane.

Theorem 3.4.1 *Suppose that the hypotheses* (h_1') - (h_4) *hold. Then the problem*

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega) \\ Lu = 0\,, \end{cases} \quad (3.14)$$

has only the zero solution.

Proof $-$ The proof is similar to that given in 2.3.1, taking into account to apply Theorem 5.2 in [17] in place of Theorem 4.3 in [11]. □

Lemma 3.4.2 *Assume that (h_4) is true. Then the Dirichlet problem*

$$\begin{cases} u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)\,, \\ -\Delta u + cu = f\,, \quad f \in L^p_s(\Omega)\,, \end{cases} \tag{3.15}$$

where

$$c = 1 + \left| -s(s+1)\sum_{i=1}^{2}\frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^{2}\frac{\sigma_{x_i x_i}}{\sigma}\right|, \tag{3.16}$$

is uniquely solvable.

Proof − Note that u is a solution of the problem (3.15) if and only if $w = \sigma^s u$ is a solution of the problem

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \\ -\displaystyle\sum_{i=1}^{2}(\sigma^{-s}w)_{x_i x_i} + c\sigma^{-s}w = f\,, \quad f \in L^p_s(\Omega)\,. \end{cases} \tag{3.17}$$

Since, for any $i \in \{1,2\}$

$$\begin{aligned} (\sigma^{-s}w)_{x_i x_i} &= \sigma^{-s}w_{x_i x_i} - 2s\sigma^{-s-1}\sigma_{x_i}w_{x_i} + s(s+1)\sigma^{-s-2}\sigma_{x_i}^2 w + \\ &- s\sigma^{-s-1}\sigma_{x_i x_i}w\,, \end{aligned}$$

then (3.17) is equivalent to the problem

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \\ -\Delta w + \displaystyle\sum_{i=1}^{n}\alpha_i w_{x_i} + \alpha w = \sigma^s f\,, \end{cases} \tag{3.18}$$

where

$$\alpha_i = 2s\frac{\sigma_{x_i}}{\sigma}\,, \qquad i = 1, 2\,,$$

$$\alpha = c - s(s+1)\sum_{i=1}^{2}\frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^{2}\frac{\sigma_{x_i x_i}}{\sigma}\;.$$

By Theorem 5.2 of [17], (1.6) of [50] and (1.24), we obtain that (3.18) is uniquely solvable and then the problem (3.15) is uniquely solvable too.

$\square$

The obtained results up to here allow to prove the existence and uniqueness theorem for the solution of the Dirichlet problem in the plane.

Theorem 3.4.3 *Suppose that the conditions (h_1') - (h_4) hold. Then the problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)\,, \\ Lu = f\,, \quad f \in L_s^p(\Omega)\,, \end{cases} \tag{3.19}$$

is uniquely solvable.

Proof $-$ For each $\tau \in [0,1]$ we put

$$L_\tau = \tau L + (1-\tau)(-\Delta + c)\,,$$

where c is the function defined by (3.16). From Theorem 1.5.1 the operator

$$\tau \in [0,1] \longmapsto L_\tau \in B(W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega), L_s^p(\Omega))$$

is continuous. By Theorem 3.3.1 we can say that the operator L_τ has closed range and by Theorem 3.4.1 it has the kernel null. Then, applying

Lemma 4.1 of [11], there exists a positive real number c_1 such that

$$||u||_{W^{2,p}_s(\Omega)} \leq c_1 ||L_\tau u||_{L^p_s(\Omega)}\,, \tag{3.20}$$
$$\forall u \in W^{2,p}_s(\Omega) \cap \overset{\circ}{W}{}^{1,p}_s(\Omega)\,, \quad \forall \tau \in [0,1]\,.$$

Therefore, Lemma 3.4.2 and the estimate (3.20) allow to use the method of continuity along a parameter (see, e.g., Theorem 5.2 of [23]) in order to prove that the problem (3.19) is uniquely solvable. $\square$

Chapter 4

The Dirichlet problem in $\mathcal{C}^2(\overline{\Omega})$ - weighted Sobolev spaces

In this chapter, we obtain some a priori bounds in $W^{2,2}$ space for a class of uniformly elliptic second order differential operators, before in a no weighted case after in a $\mathcal{C}^2(\overline{\Omega})$ weighted case. We deduce a uniqueness and existence theorem for the associated Dirichlet weighted problem on unbounded domains of $\mathbb{R}^n, n \geq 2$,

$$\begin{cases} u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega)\,, \\ Lu = f\,, \quad f \in L^2_s(\Omega)\,, \end{cases} \tag{4.1}$$

where $s \in \mathbb{R}$, $W^{2,2}_s(\Omega)$, $\overset{\circ}{W}{}^{1,2}_s(\Omega)$ and $L^2_s(\Omega)$ are weighted Sobolev spaces where the weight ρ^s is power of a function $\rho : \bar{\Omega} \to \mathbb{R}_+$, of class $\mathcal{C}^2(\bar{\Omega})$.

4.1 A no weighted a priori bound

We want to prove a $W^{2,2}$-bound for an uniformly elliptic second order linear differential operator.

Let us start proving an useful lemma. For reader's convenience, we recall here some results proved in [14], adapted to our needs.

Lemma 4.1.1 *If Ω is an open subset of $\mathbb{R}^n$ having the cone property and $g \in M^{r,\lambda}(\Omega)$, with $r > 2$ and $\lambda = 0$ if $n = 2$, and $r \in]2, n]$ and $\lambda = n - r$ if $n > 2$, then*

$$u \longrightarrow g\,u \tag{4.2}$$

is a bounded operator from $W^{1,2}(\Omega)$ to $L^2(\Omega)$. Moreover, there exists a constant $c \in \mathbb{R}_+$, such that

$$\|g\,u\|_{L^2(\Omega)} \leq c\,\|g\|_{M^{r,\lambda}(\Omega)}\,\|u\|_{W^{1,2}(\Omega)}\,, \tag{4.3}$$

with $c = c\,(\Omega, n, r)$.
Furthermore, if $g \in \widetilde{M}^{r,\lambda}(\Omega)$, then for any $\varepsilon > 0$ there exists a constant $c_\varepsilon \in \mathbb{R}_+$, such that

$$\|g\,u\|_{L^2(\Omega)} \leq \varepsilon\,\|u\|_{W^{1,2}(\Omega)} \,+ c_\varepsilon \|u\|_{L^2(\Omega)}, \tag{4.4}$$

with $c_\varepsilon = c_\varepsilon(\varepsilon, \Omega, n, r, \widetilde{\sigma}^{\,r,\lambda}[\,g])$. If $g \in M^{\,t,\mu}(\Omega)$, with $t \geq 2$ and $\mu > n-2t$, then the operator in (4.2) is bounded from $W^{2,2}(\Omega)$ to $L^2(\Omega)$. Moreover, *there exists a constant* $c' \in \mathbb{R}_+$*, such that*

$$\|g\,u\|_{L^2(\Omega)} \leq c'\, \|g\|_{M^{t,\mu}(\Omega)}\, \|u\|_{W^{2,2}(\Omega)}\,, \tag{4.5}$$

with $c' = c'(\Omega, n, t,\, \mu)$.
Furthermore, if $g \in \widetilde{M}^{t,\mu}(\Omega)$*, then for any* $\varepsilon > 0$ *there exists a constant* $c'_\varepsilon \in \mathbb{R}_+$*, such that*

$$\|g\,u\|_{L^2(\Omega)} \leq \varepsilon\, \|u\|_{W^{2,2}(\Omega)} \, + c'_\varepsilon \|u\|_{L^2(\Omega)}, \tag{4.6}$$

with $c'_\varepsilon = c'_\varepsilon(\varepsilon, \Omega, n, t,\, \mu, \widetilde{\sigma}^{t,\mu}[\,g])$.

Proof − The proof easily follows from Theorem 3.2 and Corollary 3.3 of [14]. □

From now on we assume that Ω is an unbounded open subset of $\mathbb{R}^n, n \geq 2$, with the uniform $C^{1,1}$-regularity property.

We consider the differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij}\, \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} a_i\, \frac{\partial}{\partial x_i} + a\,, \tag{4.7}$$

with the following conditions on the coefficients:

$$(h_1) \qquad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega), \;\; i,j = 1, \ldots, n\,, \\ \exists\, \nu > 0 \;\; : \;\; \displaystyle\sum_{i,j=1}^{n} a_{ij}\, \xi_i\, \xi_j \;\geq\; \nu|\xi|^2 \;\; \text{a.e. in } \; \Omega\,,\; \forall \xi \in \mathbb{R}^n\,, \end{cases}$$

$$(h_2) \qquad \begin{cases} (a_{ij})_{x_j}, a_i \in M_o^{r,\lambda}(\Omega)\,, \quad i,j=1,\dots,n\,, \\ \text{with } r>2 \text{ and } \lambda=0 \text{ if } n=2\,, \\ \text{with } r\in]2,n] \text{ and } \lambda=n-r \text{ if } n>2\,, \end{cases}$$

$$(h_3) \qquad \begin{cases} a\in \widetilde{M}^{\,t,\mu}(\Omega)\,, \text{ with } t\geq 2 \text{ and } \mu>n-2t\,, \\ \operatorname*{ess\,inf}_{\Omega} a=a_0>0. \end{cases}$$

We explictly observe that under the assumptions (h_1) - (h_3) the operator $L: W^{2,2}(\Omega)\to L^2(\Omega)$ is bounded, as a consequence of Lemma 4.1.1. We are now in position to prove the above mentioned a priori estimate.

Theorem 4.1.2 *Let L be defined in (4.7). Under hypotheses (h_1)-(h_3), there exists a constant $c\in\mathbb{R}_+$ such that*

$$\|u\|_{W^{2,2}(\Omega)}\leq c\|Lu\|_{L^2(\Omega)}, \quad \forall\, u\in W^{2,2}(\Omega)\cap \overset{\circ}{W}{}^{1,2}(\Omega)\,, \tag{4.8}$$

with $c=c(\Omega,n,\nu,r,t,\mu,||a_{ij}||_{L^\infty(\Omega)},\sigma_o^{r,\lambda}[(a_{ij})_{x_j}],\sigma_o^{r,\lambda}[a_i],\widetilde{\sigma}^{t,\mu}[a],a_0)$.

Proof $-$ Let us put

$$L_0=-\sum_{i,j=1}^n a_{ij}\,\frac{\partial^2}{\partial x_i\,\partial x_j}$$

and fix $u\in W^{2,2}(\Omega)\cap \overset{\circ}{W}{}^{1,2}(\Omega)$. Lemma 1 being true, Lemma 3.1 of [17] (for $n=2$) and Theorem 5.1 of [14] (for $n>2$) apply, so that there exists a constant $c_1 \in \mathbb{R}_+$ such that

$$\|u\|_{W^{2,2}(\Omega)} \leq c_1(\|L_0 u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}),$$

with $c_1 = c_1(\Omega, n, \nu, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{r,\lambda}[(a_{ij})_{x_j}])$. Therefore,

$$\begin{gathered}\|u\|_{W^{2,2}(\Omega)} \leq c_1(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \\ \textstyle\sum_{i=1}^n \|a_i u_{x_i}\|_{L^2(\Omega)} + \|au\|_{L^2(\Omega)}).\end{gathered} \tag{4.9}$$

On the other hand, from Lemma 4.1.1 one has

$$\begin{cases} \|a_i u_{x_i}\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{W^{2,2}(\Omega)} + c_\varepsilon \|u_{x_i}\|_{L^2(\Omega)}, \\ \\ \|au\|_{L^2(\Omega)} \leq \varepsilon \|u\|_{W^{2,2}(\Omega)} + c'_\varepsilon \|u\|_{L^2(\Omega)}, \end{cases} \tag{4.10}$$

with $c_\varepsilon = c_\varepsilon(\varepsilon, \Omega, n, r, \sigma_o^{r,\lambda}[a_i])$ and $c'_\varepsilon = c'_\varepsilon(\varepsilon, \Omega, n, t, \mu, \widetilde{\sigma}^{t,\mu}[a])$. Furthermore, classical interpolation results give that there exists a constant $K \in \mathbb{R}_+$ such that

$$\|u_x\|_{L^2(\Omega)} \leq K\varepsilon \|u\|_{W^{2,2}(\Omega)} + \frac{K}{\varepsilon} \|u\|_{L^2(\Omega)}, \tag{4.11}$$

with $K = K(\Omega)$. Combining (4.9), (4.10) and (4.11) we conclude that there exists $c_2 \in \mathbb{R}_+$ such that

$$\|u\|_{W^{2,2}(\Omega)} \leq c_2(\|Lu\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{4.12}$$

with $c_2 = c_2(\Omega, n, \nu, r, t, \mu, ||a_{ij}||_{L^\infty(\Omega)}, \sigma_o^{r,\lambda}[(a_{ij})_{x_j}], \sigma_o^{r,\lambda}[a_i], \widetilde{\sigma}^{t,\mu}[a])$.

To show (4.8) it remains to estimate $\|u\|_{L^2(\Omega)}$. To this aim let us rewrite our operator in divergence form

$$Lu = -\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j} + \sum_{i=1}^{n}\big(\sum_{j=1}^{n}(a_{ij})_{x_j} + a_i\big)u_{x_i} + au\,, \tag{4.13}$$

in order to adapt to our framework some known results concerning operators in variational form. Following along the lines the proofs of Theorem 4.3 of [49] (for $n = 2$) and of Theorem 4.2 of [52] (for $n > 2$), with opportune modifications - we explicitly observe that the continuity of the bilinear form associated to (4.13) in our case is an immediate consequence of Lemma 4.1.1 - we obtain that

$$\|u\|_{L^2(\Omega)} \leq c_3\|Lu\|_{L^2(\Omega)}, \tag{4.14}$$

with $c_3 = c_3(n, \nu, r, \sigma_o^{r,\lambda}[(a_{ij})_{x_j}], \sigma_o^{r,\lambda}[a_i], a_0)$. Putting together (4.12) and (4.14) we obtain (4.8).

The $W^{2,2}$-bound obtained in Theorem 4.8 allows us to show an a priori estimate in the weighted case. At this aim, let us introduce the following

4.2 Preliminary results

Let us consider the class of $\mathcal{C}^k(\bar{\Omega})$ - weight functions, as in section 1.3, with $k = 2$. Let be a weight $\rho : \bar{\Omega} \to \mathbb{R}_+$, $\rho \in C^2(\bar{\Omega})$ and such that (1.9) is satisfied (for $k = 2$). Moreover, we assume that

$$\lim_{|x| \to +\infty} \left(\rho(x) + \frac{1}{\rho(x)} \right) = +\infty \quad \text{and} \quad \lim_{|x| \to +\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0. \quad (4.15)$$

An example of a function verifying our hypotheses is given by

$$\rho(x) = (1 + |x|^2)^t, \quad t \in \mathbb{R} \backslash \{0\}.$$

We associate to ρ a function σ defined by

$$\begin{cases} \sigma = \rho & \text{if } \rho \to +\infty \quad \text{for } |x| \to +\infty, \\ \\ \sigma = \dfrac{1}{\rho} & \text{if } \rho \to 0 \quad \text{for } |x| \to +\infty. \end{cases} \quad (4.16)$$

Clearly σ verifies (1.9) and

$$\lim_{|x| \to +\infty} \sigma(x) = +\infty, \quad \lim_{|x| \to +\infty} \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0. \quad (4.17)$$

Now, let us fix a cutoff function $f \in C_\circ^\infty(\bar{\mathbb{R}}_+)$ such that

$$0 \leq f \leq 1, \quad f(t) = 1 \text{ if } t \in [0, 1], \quad f(t) = 0 \text{ if } t \in [2, +\infty[.$$

Then, set

$$\zeta_k : x \in \bar{\Omega} \longrightarrow f\left(\frac{\sigma(x)}{k}\right), \quad k \in \mathbb{N}$$

and

$$\Omega_k = \{x \in \Omega \;:\; \sigma(x) < k\}, \quad k \in \mathbb{N}. \tag{4.18}$$

By our definition it follows that $\zeta_k \in C_\circ^\infty(\bar{\Omega})$ and

$$0 \leq \zeta_k \leq 1, \quad \zeta_k = 1 \text{ on } \overline{\Omega}_k, \quad \zeta_k = 0 \text{ on } \overline{\Omega \setminus \Omega_{2k}}, \quad k \in \mathbb{N}.$$

Finally, we introduce the sequence

$$\eta_k : x \in \bar{\Omega} \longrightarrow 2k\,\zeta_k(x) + (1 - \zeta_k(x))\sigma(x), \quad k \in \mathbb{N}.$$

For any $k \in \mathbb{N}$, one has

$$\begin{aligned}
\eta_k &= \zeta_k(2k - \sigma) + \sigma \geq \sigma && \text{in } \overline{\Omega_{2k}}, && (4.19)\\
\eta_k &\leq 2k + \sigma \leq \Big(\frac{2k}{\inf_{\overline{\Omega_{2k}}} \sigma} + 1\Big)\sigma = (c_k + 1)\sigma && \text{in } \overline{\Omega_{2k}}, && (4.20)\\
\eta_k &= \sigma && \text{in } \overline{\Omega \setminus \Omega_{2k}}, && (4.21)
\end{aligned}$$

where $c_k \in \mathbb{R}_+$ depends only on k. This entails that

$$\sigma \sim \eta_k, \qquad \forall k \in \mathbb{N}. \tag{4.22}$$

Concerning the derivatives, easy calculations give that, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
(\eta_k)_x &= (\eta_k)_{xx} = 0 && \text{in } \overline{\Omega_k}, && (4.23)\\
(\eta_k)_x &= \sigma_x, \qquad (\eta_k)_{xx} = \sigma_{xx} && \text{in } \overline{\Omega \setminus \Omega_{2k}}, && (4.24)\\
(\eta_k)_x &\le c_1 \sigma_x, \quad (\eta_k)_{xx} \le c_2 \left(\frac{\sigma_x^2}{\sigma} + \sigma_{xx} \right) && \text{in } \overline{\Omega_{2k} \setminus \Omega_k}, && (4.25)
\end{aligned}
$$

with c_1 and c_2 positive constants independent of x and k.

From (4.19), (4.21), (4.23), (4.24) and (4.25), we obtain, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
\frac{(\eta_k)_x}{\eta_k} &\le c_1' \frac{\sigma_x}{\sigma} && \text{in } \overline{\Omega}, && (4.26)\\
\frac{(\eta_k)_{xx}}{\eta_k} &\le c_2' \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} && \text{in } \overline{\Omega}, && (4.27)
\end{aligned}
$$

where c_1' and c_2' are positive constants independent of x and k.

Combining (4.23), (4.26) and (4.27) we have also, for any $k \in \mathbb{N}$,

$$
\begin{aligned}
\frac{(\eta_k)_x}{\eta_k} &\le c_1' \sup_{\overline{\Omega \setminus \Omega_k}} \frac{\sigma_x}{\sigma} && \text{in } \overline{\Omega}, && (4.28)\\
\frac{(\eta_k)_{xx}}{\eta_k} &\le c_2' \sup_{\overline{\Omega \setminus \Omega_k}} \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} && \text{in } \overline{\Omega}. && (4.29)
\end{aligned}
$$

We conclude this section proving the following lemma:

Lemma 4.2.1 *Let σ and Ω_k, $k \in \mathbb{N}$, be defined by* (4.16) *and* (4.18), *respectively. Then*

$$\lim_{k\to+\infty}\sup_{\Omega\setminus\Omega_k}\frac{\sigma_x(x)+\sigma_{xx}(x)}{\sigma(x)}=0. \tag{4.30}$$

Proof – Set

$$\varphi(x)=\frac{\sigma_x(x)+\sigma_{xx}(x)}{\sigma(x)},\quad x\in\bar{\Omega}$$

and

$$\psi_k=\sup_{\overline{\Omega\setminus\Omega_k}}\varphi,\quad k\in\mathbb{N}.$$

By the second relation in (4.17) the supremum of φ over $\overline{\Omega\setminus\Omega_k}$ is actually a maximum, thus, for every $k\in\mathbb{N}$, there exists $x_k\in\overline{\Omega\setminus\Omega_k}$ such that $\psi_k=\varphi(x_k)$.

To prove (4.30) we have to show that $\lim_{k\to+\infty}\psi_k=0$.

We proceed by contradiction. Hence, let us assume that there exists $\varepsilon_0>0$ such that, for any $k\in\mathbb{N}$, there exists $n_k>k$ such that $\psi_{n_k}=\varphi(x_{n_k})\geq\varepsilon_0$.

If the sequence $(x_{n_k})_{k\in\mathbb{N}}$ is bounded, there exists a subsequence $(x'_{n_k})_{k\in\mathbb{N}}$ converging to a limit $x\in\bar{\Omega}$, and by the continuity of σ, $(\sigma(x'_{n_k}))_{k\in\mathbb{N}}$ converges to $\sigma(x)$. On the other hand, $x'_{n_k}\in\overline{\Omega\setminus\Omega_k}$, thus $\sigma(x'_{n_k})\geq n_k$, which is in contrast with the fact that $(\sigma(x'_{n_k}))_{k\in\mathbb{N}}$ is a convergent sequence.

Therefore $(x_{n_k})_{k\in\mathbb{N}}$ is unbounded, so that there exists a subsequence $(x''_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty}|x''_{n_k}|=+\infty$. Thus, by the second relation in (4.17) one has $\lim_{k\to+\infty}\varphi(x''_{n_k})=0$. This gives the contradiction since $\varphi(x''_{n_k})\geq\varepsilon_0$. □

4.3 A weighted a priori bound

Now, we are in the position to state a $W^{2,2}_s(\bar{\Omega})$- a priori bound for an uniformly elliptic second order linear differential operator.

Theorem 4.3.1 *Let L be defined in (4.7). Under hypotheses (h_1)-(h_3), there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W^{2,2}_s(\Omega)} \leq c\|Lu\|_{L^2_s(\Omega)}, \quad \forall\, u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega)\,, \tag{4.31}$$

with $c = c(\Omega, n, s, \nu, r, t, \mu, ||a_{ij}||_{L^\infty(\Omega)}, ||a_i||_{M^{r,\lambda}(\Omega)}, {\sigma_o}^{r,\lambda}[(a_{ij})_{x_j}], {\sigma_o}^{r,\lambda}[a_i], \widetilde{\sigma}^{t,\mu}[a], a_0)$.

Proof − Fix $u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega)$. In the sequel, for sake of simplicity, we will write $\eta_k = \eta$, for a fixed $k \in \mathbb{N}$. Observe that η satisfies (1.9), as a consequence of (4.26) and (4.27), so that Lemma 1.3.6 applies giving that $\eta^s u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$. Therefore, in view of Theorem 4.1.2, there exists $c_0 \in \mathbb{R}_+$, such that

$$||\eta^s\, u||_{W^{2,2}(\Omega)} \leq c_0||L(\eta^s u)||_{L^2(\Omega)}, \tag{4.32}$$

with $c_0 = c_0(\Omega, n, \nu, r, t, \mu, ||a_{ij}||_{L^\infty(\Omega)}, {\sigma_o}^{r,\lambda}[(a_{ij})_{x_j}], {\sigma_o}^{r,\lambda}[a_i], \widetilde{\sigma}^{t,\mu}[a], a_0)$.
Easy computations give

$$\begin{aligned} L(\eta^s u) &= \eta^s Lu - s\sum_{i,j=1}^n a_{ij}\Big((s-1)\eta^{s-2}\eta_{x_i}\eta_{x_j}u + \eta^{s-1}\eta_{x_ix_j}u + \\ &+ 2\eta^{s-1}\eta_{x_i}u_{x_j}\Big) + s\sum_{i=1}^n a_i\eta^{s-1}\eta_{x_i}u. \end{aligned} \tag{4.33}$$

Putting together (4.32) and (4.33) we deduce that

$$\begin{aligned} ||\eta^s\, u||_{W^{2,2}(\Omega)} &\le c_1\Big(||\eta^s\, Lu||_{L^2(\Omega)} + \sum_{i,j=1}^n \big(||\eta^{s-2}\eta_{x_i}\eta_{x_j}u||_{L^2(\Omega)} + \\ &+ ||\eta^{s-1}\eta_{x_ix_j}u||_{L^2(\Omega)} + ||\eta^{s-1}\eta_{x_i}u_{x_j}||_{L^2(\Omega)}\big) + \\ &+ \sum_{i=1}^n ||a_i\eta^{s-1}\eta_{x_i}u||_{L^2(\Omega)}\Big), \end{aligned} \tag{4.34}$$

where $c_1 \in \mathbb{R}_+$ depends on the same parameters as c_0 and on s.
On the other hand, from Lemma 4.1.1 and (4.28) we get

$$||a_i\eta^{s-1}\eta_{x_i}u||_{L^2(\Omega)} \le c_2 \sup_{\Omega\setminus\Omega_k}\frac{\sigma_x}{\sigma}\ ||a_i||_{M^{r,\lambda}(\Omega)}||\eta^s u||_{W^{1,2}(\Omega)}, \tag{4.35}$$

with $c_2 = c_2(\Omega, n, r)$. Combining (4.28), (4.29), (4.34) and (4.35), with simple calculations we obtain the bound

$$\begin{aligned} ||\eta^s u||_{W^{2,2}(\Omega)} &\leq c_3 \Bigg[||\eta^s Lu||_{L^2(\Omega)} + \Bigg(\sup_{\Omega \setminus \Omega_k} \frac{\sigma_x^2 + \sigma\, \sigma_{xx}}{\sigma^2} + \\ &+ \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x}{\sigma} \Bigg) \, ||\eta^s u||_{W^{2,2}(\Omega)} \Bigg], \end{aligned} \tag{4.36}$$

where c_3 depends on the same parameters as c_1 and on $\|a_i\|_{M^{r,\lambda}(\Omega)}$.
By Lemma 4.2.1, it follows that there exists $k_o \in \mathbb{N}$ such that

$$\left(\sup_{\Omega \setminus \Omega_{k_o}} \frac{\sigma_x^2 + \sigma\, \sigma_{xx}}{\sigma^2} + \sup_{\Omega \setminus \Omega_{k_o}} \frac{\sigma_x}{\sigma} \right) \leq \frac{1}{2\, c_3}. \tag{4.37}$$

Now, if we still denote by η the function η_{k_o}, from (4.36) and (4.37) we deduce that

$$||\eta^s u||_{W^{2,2}(\Omega)} \leq 2c_3 ||\eta^s Lu||_{L^2(\Omega)}. \tag{4.38}$$

Then, by Lemma 1.3.3 and by (4.22), written for $k = k_o$, we have

$$\sum_{|\alpha| \leq 2} ||\sigma^s \partial^\alpha u||_{L^2(\Omega)} \leq c_4 ||\sigma^s Lu||_{L^2(\Omega)}, \tag{4.39}$$

with c_4 depending on the same parameters as c_3 and on k_o.
This last estimate being true for every $s \in \mathbb{R}$, we also have

$$\sum_{|\alpha| \leq 2} ||\sigma^{-s} \partial^\alpha u||_{L^2(\Omega)} \leq c_5 ||\sigma^{-s} Lu||_{L^2(\Omega)}. \tag{4.40}$$

The bounds in (4.39) and (4.40) together with the definition (4.16) of σ, give estimate (4.3.1). $\square$

4.4 Uniqueness and existence results

This section is devoted to the proof of the solvability of the Dirichlet problem (4.1).

Lemma 4.4.1 *The Dirichlet problem*

$$\begin{cases} u \in W_s^{2,2}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega)\,, \\ \\ -\Delta u + bu = f\,, \quad f \in L_s^2(\Omega)\,, \end{cases} \tag{4.41}$$

where

$$b = 1 + \left| - s(s+1)\sum_{i=1}^{n} \frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^{n} \frac{\sigma_{x_i x_i}}{\sigma} \right|,$$

is uniquely solvable.

Proof $-$ Observe that u is a solution of problem (4.41) if and only if $w = \sigma^s u$ is a solution of the problem

$$\begin{cases} w \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)\,, \\ \\ -\Delta(\sigma^{-s} w) + b\sigma^{-s} w = f\,, \quad f \in L_s^2(\Omega)\,. \end{cases} \tag{4.42}$$

Clearly, for any $i \in \{1, \cdots, n\}$,

$$\frac{\partial^2}{\partial x_i^2}(\sigma^{-s} w) = \sigma^{-s} w_{x_i x_i} - 2s\sigma^{-s-1}\sigma_{x_i} w_{x_i} + s(s+1)\sigma^{-s-2}\sigma_{x_i}^2 w - s\sigma^{-s-1}\sigma_{x_i x_i} w,$$

then (4.42) is equivalent to the problem

$$\begin{cases} w \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)\,, \\ \\ -\Delta w + \sum_{i=1}^{n} \alpha_i w_{x_i} + \alpha w = g\,, \quad g \in L^2(\Omega)\,, \end{cases} \tag{4.43}$$

where

$$\alpha_i = 2s\frac{\sigma_{x_i}}{\sigma}\,, i = 1, \cdots, n,\ \alpha = b - s(s+1)\sum_{i=1}^{n} \frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^{n} \frac{\sigma_{x_i x_i}}{\sigma}\,,\ g = \sigma^s f\,.$$

Using Theorem 5.2 in [17] (for $n = 2$), Theorem 4.3 of [11] (for $n > 2$), (1.6) of [50] and the hypotheses on σ, we obtain that (4.43) is uniquely solvable and then problem (4.41) is uniquely solvable too. □

Theorem 4.4.2 *Let L be defined in* (4.7)*. Under hypotheses* (h_1) – (h_3)*, the problem*

$$\begin{cases} u \in W_s^{2,2}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega)\,, \\ \\ Lu = f\,, \quad f \in L_s^2(\Omega)\,, \end{cases} \tag{4.44}$$

is uniquely solvable.

Proof − For each $\tau \in [0,1]$ we put

$$L_\tau = \tau(L) + (1-\tau)(-\Delta + b)\,.$$

In view of Theorem 4.3.1

$$||u||_{W^{2,2}_s(\Omega)} \leq c||L_\tau u||_{L^p_s(\Omega)},$$

$$\forall u \in W^{2,2}_s(\Omega) \cap \overset{\circ}{W}{}^{1,2}_s(\Omega)\,,\ \forall \tau \in [0,1]\,.$$

Thus, taking into account the result of Lemma 4.4.1 and using the method of continuity along a parameter (see, e.g., Theorem 5.2 of [23]), we obtain the claimed result. □

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